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Two-loop QCD helicity amplitudes for massless quark-massless gauge boson scattering

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Abstract: We present the two-loop helicity amplitudes for the scattering of massless quarks and massless gauge bosons in QCD. We use projector techniques to compute the coefficients of the general tensor describing the two-quark two-boson amplitude in conventional dimensional regularisation and use these coefficients to derive the helicity amplitudes in the 't Hooft-Veltman scheme. The structure of the infrared divergences agrees with that predicted by Catani while expressions for the finite remainders for $q\bar{q} \to gg$, $q\bar{q} \to g\gamma$, $q\bar{q} \to \gamma\gamma$ and the processes related by crossing symmetry are given in terms of logarithms and polylogarithms that are real in the physical region. We have checked that the interference of tree and two-loop helicity amplitudes, summed over helicities and colours, reproduces the previous results for the finite remainders for interference of tree and two-loop amplitudes given in refs. [1, 2]. We also find agreement with the two-loop helicity amplitudes for quark-gluon scattering presented in ref. [3].

Keywords: QCD, NLO Computations, Jets, Hadronic Colliders.

Contents

1. Introduction

QCD perturbation theory has proven to be a reliable tool with which to generate theoretical predictions of hard scattering processes that can be compared with experimental results from electron-positron annihilation, electron-proton and hadron-hadron scattering. In such studies, the theoretical analysis of an observable at next-to-leading order (NLO) in α_s has become a *standard candle*, the best theoretical prediction to compare against experiment. These NLO calculations are also used to provide an estimate of the uncalculated theoretical uncertainties via the variation in the renormalisation (and factorisation) scale.

However, the large amounts of data soon to become available from for example, high-energy hadronic collider runs place a demand for more accurate theoretical predictions. To go beyond NLO to the next order in QCD perturbation theory, next-to-next-to-leading order (NNLO), has a direct impact on several aspects of the theoretical predictions. For example [4], the renormalisation (and factorization) scale uncertainty is significantly diminished, there is a better matching of the parton level jet algorithm used in theory and the hadron level jet algorithm used by experiment and the contributions attributed to non-perturbative power corections may also be reduced.

There are several ingredients that need to be considered to calculate the NNLO corrections for a specific $2 \to 2$ scattering process,

- tree-level $2 \to 4$ partonic amplitudes
- one-loop $2 \to 3$ partonic amplitudes,
- two-loop $2 \to 2$ partonic amplitudes.

The tree-level six-point amplitudes [5]–[12] and one-loop five point amplitudes [13, 14, 15] have been known for some time while there has been recent progress in computing two-loop amplitudes. Parton distribution functions from a global fit to available data with observables computed consistently at NNLO [4] coupled with a complete knowledge of the three-loop splitting functions [16]–[24] to generate the evolution of the parton distribution functions at this order are also required.

In this particular paper we are concerned with processes involving two massless quarks and two massless gauge bosons,

$$q\bar{q} \to gg\,, \qquad q\bar{q} \to g\gamma\,, \qquad q\bar{q} \to \gamma\gamma$$

and the processes related by time reversal and crossing symmetry. In previous work [1, 2], we computed both the self-interference of the one-loop amplitudes and the interference of the tree and two-loop amplitudes for these processes in conventional dimensional regularisation (CDR). In that work, the matrix-elements were averaged over all external helicities and summed over colours in the high energy limit where all quark masses can be neglected. Here, we extend these calculations to now compute the helicity amplitudes for these processes.

The one-loop helicity amplitudes for parton-parton scattering processes have been known for some time [25]. More recently, there have been a few two-loop helicity amplitudes calculated within CDR with $d = 4 - 2\epsilon$ dimensions:

- for the $gg \to \gamma\gamma$ [26], $\gamma\gamma \to \gamma\gamma$ [27, 28] and $gg \to gg$ [29, 30] scattering processes, the explicit contraction of the amplitude with the external polarization vectors was used.
- for the for the $\gamma\gamma \to \gamma\gamma$ [28] and $e^+e^- \to 3$ jets [31] scattering processes, the individual components of the Lorentz-invariant amplitude decomposition were projected out.

After the explicit helicity structure is obtained via any of these methods, one needs to tackle a large number of two-loop integrals with different topologies and tensor structure. It is now well known (see for example [32, 33, 34] and references therein), that these integrals can be reduced [35] to a handful of *master integrals* using integration-by-parts [36, 37] and Lorentz-invariance [38] identities. For the process at hand, the relevant master integrals are 10 massless four-point functions with all legs on-shell which have been calculated in the past few years using a wide variety of methods [39]–[48].

In this paper, we follow a similar appoach to extract the helicity structure of the amplitude as the one outlined in refs. [28, 31]. We start with a general tensor that describes the two quark-two boson amplitude and use gauge invariance to find relations among the tensor coefficients. These coefficients can be extracted using a set of projectors that are valid at any order in perturbation theory. The projectors saturate the tensor structure and produce scalar integrals. The methodology of reducing these scalar integrals to master integrals is exactly the same as for the calculation of the spin summed matrix elements [1, 2]. Note that the tensor coefficients are evaluated in CDR.

Then, we provide the perturbative expansion for the one- and two-loop tensor coefficients and remove ultraviolet (UV) divergences at each order in α_s by renormalisation within the $\overline{\rm MS}$ scheme. The infrared (IR) divergent structure is shown to agree with the one obtained in refs. [49]. Note that in extracting the coefficients of the general tensor structure, conventional dimensional regularisation is preserved.

Finally, the helicity amplitudes are obtained from the general tensor using standard helicity techniques and within the t'Hooft-Veltman (HV) scheme where the external states are four-dimensional and the internal states are kept in 4-2ϵ-dimensions [50, 51]. The finite helicity amplitudes are the main new results presented in this paper and we give explicit analytic expressions valid for each helicity configuration and process in terms of logarithms and polylogarithms that are real in the physical domain.

During the writing of this paper, we became aware of an independent calculation of the two-loop helicity amplitudes for quark-gluon scattering [3]. This paper [3] uses a rather different method to extract the helicity amplitudes in a variety of schemes. Where the two calculations overlap (in the HV scheme), we have compared the finite remainder terms for the two-loop $q\bar{q} \to gg$ processes and find complete agreement.

The paper is organised as follows. In section 2 we introduce notation and discuss the general tensor structure of the two quark-two boson amplitude. Gauge invariance is used to reduce the number of independent tensor structures. We give projectors to isolate the various coefficients and relate the individual coefficients to the interference of amplitudes. Section 3 shows the perturbative expansion for the tensor coefficients computed within CDR and presents an analysis of their UV and IR divergent structure. The coefficients are renormalised in the $\overline{\rm MS}$ scheme to remove all UV divergences, while the IR poles are predicted using the results of ref. [49]. Expressions for the one-loop tensor coefficients to all orders in ϵ appear in section 3.2 in terms of one-loop bubble integral and the one-loop box integral in $d = 6 - 2\epsilon$. The relationship between the tensor coefficients and the helicity amplitudes in the HV scheme as described in section 4. The UV and IR structure of the helicity amplitudes is exactly the same as for the tensor coefficients, and finite $\mathcal{O}\left(\epsilon^0\right)$ remainders are obtained by subtracting the predicted pole structure from the explicit calculation both at one-loop and two-loop. The main results of this paper are the finite remainders of the two-loop helicity amplitudes. The expressions are rather lengthy and we confine ourselves to presenting the amplitudes for the $q\bar{q} \to gg$ process in appendix A. For completeness, the finite one-loop remainders are collected in appendix B. Results for the processes related by crossing symmetry and those processes involving photons can be obtained from the authors. Finally, in section 6 we present a discussion of our results and conclude.

2. Notation

The generic process we consider in this paper is

$$0 \to q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) + b_3(p_3, \lambda_3) + b_4(p_4, \lambda_4)\,, \tag{2.1}$$

where the bosons b_3 and b_4 are either gluons or photons. All particles are light-like with momenta satisfying,

$$0 \to p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu\,, \qquad p_i^2 = 0\,. \tag{2.2}$$

Physical processes are obtained by crossing particles into the initial state. The associated Mandelstam variables are given by

$$s_{12} = (p_1 + p_2)^2\,, \qquad s_{23} = (p_2 + p_3)^2\,, \qquad s_{13} = (p_1 + p_3)^2\,, \qquad s_{12} + s_{23} + s_{13} = 0\,. \tag{2.3}$$

We work in conventional dimensional regularisation treating all external quark and boson states in d dimensions and renormalise the ultraviolet divergences in the $\overline{\rm MS}$ scheme. The bare coupling α_0 is related to the running coupling $\alpha_s \equiv \alpha_s(\mu^2)$ at renormalisation scale μ, by

$$\alpha_0 \mu_0^{2\epsilon}\, S_\epsilon = \alpha_s\, \mu^{2\epsilon} \left[1 - \frac{\beta_0}{\epsilon}\left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right)\right], \tag{2.4}$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma}\,, \qquad \gamma = 0.5772\ldots = \text{Euler constant} \tag{2.5}$$

is the typical phase-space volume factor in $d = 4 - 2\epsilon$ dimensions and μ_0^2 is the mass parameter introduced in dimensional regularisation [52, 53, 50, 51] to maintain a dimensionless coupling in the bare QCD lagrangian density.

The first two coefficients of the QCD beta function, β_0 and β_1, for N_F (massless) quark flavours, are

$$\beta_0 = \frac{11 C_A - 4 T_R N_F}{6}\,, \qquad \beta_1 = \frac{17 C_A^2 - 10 C_A T_R N_F - 6 C_F T_R N_F}{6}\,, \tag{2.6}$$

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where N is the number of colours and

$$C_F = \frac{N^2-1}{2N}\,, \qquad C_A = N\,, \qquad T_R = \frac{1}{2}\,. \tag{2.7}$$

2.1 The general tensor

The most general tensor structure for the amplitude, $|\mathcal{M}\rangle$, is

$$\begin{aligned} |\mathcal{M}\rangle = &\ \bar u(p_1)\not{p}_3 u(p_2)\left(A_{11}\epsilon_3.p_1\ \epsilon_4.p_1 + A_{12}\epsilon_3.p_1\ \epsilon_4.p_2\right) + \\ &+\bar u(p_1)\not{p}_3 u(p_2)\left(A_{21}\epsilon_3.p_2\ \epsilon_4.p_1 + A_{22}\epsilon_3.p_2\ \epsilon_4.p_2\right) + \\ &+\bar u(p_1)\not{\epsilon}_4 u(p_2)\left(B_1\epsilon_3.p_1 + B_2\epsilon_3.p_2\right) + \\ &+\bar u(p_1)\not{\epsilon}_3 u(p_2)\left(C_1\epsilon_4.p_1 + C_2\epsilon_4.p_2\right) + \\ &+D_1\bar u(p_1)\not{\epsilon}_3\not{p}_3\not{\epsilon}_4 u(p_2) + \\ &+D_2\bar u(p_1)\not{\epsilon}_4\not{p}_3\not{\epsilon}_3 u(p_2)\,, \end{aligned} \tag{2.8}$$

where the constraints $\epsilon_3 \cdot p_3 = \epsilon_4 \cdot p_4 = 0$ have been applied. All coefficients depend on the identity of the gauge bosons and are functions of s_{13} and s_{23} (and implicitly $s_{12} = -s_{13} - s_{23}$) where $s_{ij} = (p_i + p_j)^2$. This tensor structure is a priori d-dimensional, since the dimensionality of the external states has not yet been specified. The amplitude $|\mathcal{M}\rangle$ obeys the Ward identity when one of the gauge boson polarization vectors is replaced with the corresponding boson momentum,

$$|\mathcal{M}\rangle(\epsilon_3 \to p_3) = |\mathcal{M}\rangle(\epsilon_4 \to p_4) = 0\,. \tag{2.9}$$

These constraints yield relations amongst the 10 distinct tensor structures and applying these identities gives the gauge-invariant form of the tensor,

$$\begin{aligned} |\mathcal{M}\rangle = &\ A_{11}(s_{13},s_{23})\,T_{11} + A_{12}(s_{13},s_{23})\,T_{12} + \\ &+A_{21}(s_{13},s_{23})\,T_{21} + A_{22}(s_{13},s_{23})\,T_{22} + \\ &+B(s_{13},s_{23})\,T\,, \end{aligned} \tag{2.10}$$

where A_{IJ} and B are five gauge-independent functions and the tensor structures T_{IJ} and T are given by

$$T_{1J} = \bar u(p_1)\not{p}_3 u(p_2)\epsilon_3.p_1\epsilon_4.p_J - \frac{s_{13}}{2}\bar u(p_1)\not{\epsilon}_3 u(p_2)\epsilon_4.p_J - \frac{s_{J4}}{4}\bar u(p_1)\not{\epsilon}_3\not{p}_3\not{\epsilon}_4 u(p_2)\,, \tag{2.11}$$

$$T_{2J} = \bar u(p_1)\not{p}_3 u(p_2)\epsilon_3.p_2\epsilon_4.p_J - \frac{s_{23}}{2}\bar u(p_1)\not{\epsilon}_3 u(p_2)\epsilon_4.p_J - \frac{s_{J4}}{4}\bar u(p_1)\not{\epsilon}_4\not{p}_3\not{\epsilon}_3 u(p_2)\,, \tag{2.12}$$

$$\begin{aligned} T = &\ s_{23}\left(\bar u(p_1)\not{\epsilon}_4 u(p_2)\epsilon_3.p_1 + \frac{1}{2}\bar u(p_1)\not{\epsilon}_3\not{p}_3\not{\epsilon}_4 u(p_2)\right) - \\ &-s_{13}\left(\bar u(p_1)\not{\epsilon}_4 u(p_2)\epsilon_3.p_2 + \frac{1}{2}\bar u(p_1)\not{\epsilon}_4\not{p}_3\not{\epsilon}_3 u(p_2)\right)\,, \end{aligned} \tag{2.13}$$

with $s_{14} = s_{23}$. Each of the tensor structures satisfies the Ward identities. The tensor coefficients are further related by symmetry under the interchange of the momenta (and colour labels) of the two gauge bosons,

$$\begin{aligned} A_{21}(s_{13},s_{23}) &= -A_{12}(s_{23},s_{13})\,, \\ A_{22}(s_{13},s_{23}) &= -A_{11}(s_{23},s_{13})\,, \\ B(s_{13},s_{23}) &= B(s_{23},s_{13})\,. \end{aligned} \tag{2.14}$$

2.2 Projectors for the tensor coefficients

The coefficients A_{IJ} and B may be easily extracted from a Feynman diagram calculation, using projectors such that

$$\sum_{\text{spins}} \mathcal{P}(X)\ |\mathcal{M}\rangle = X(s_{13}, s_{23})\,. \tag{2.15}$$

The explicit forms for the projectors in d space-time dimensions are,

$$\begin{aligned}
\mathcal{P}(A_{11}) \;=\; & \frac{d}{2s_{12}^2 s_{13} s_{23}(d-3)} T_{11}^\dagger + \frac{(d-4)}{2s_{12}^2 s_{13} s_{23}(d-3)} T_{22}^\dagger + \frac{1}{2s_{12}^2 s_{13} s_{23}(d-3)} T^\dagger + \\
& + \frac{(2s_{13}-(d-4)s_{23})}{2s_{12}^2 s_{13}^2 s_{23}(d-3)} T_{12}^\dagger + \frac{(2s_{23}-(d-4)s_{13})}{2s_{12}^2 s_{13} s_{23}^2(d-3)} T_{21}^\dagger\,, \\
\mathcal{P}(A_{12}) \;=\; & \frac{(2s_{13}-(d-4)s_{23})}{2s_{12}^2 s_{13}^2 s_{23}(d-3)} T_{11}^\dagger + \frac{(2s_{13}-(d-4)s_{23})}{2s_{12}^2 s_{13}^2 s_{23}(d-3)} T_{22}^\dagger - \frac{(2s_{13}-(d-4)s_{23})}{2s_{12}^2 s_{13}^2 s_{23}(d-3)(d-4)} T^\dagger + \\
& + \frac{(4(d-3)s_{13}^2 + 4(d-4)s_{13}s_{23} + d(d-4)s_{23}^2)}{2s_{12}^2 s_{13}^3 s_{23}(d-3)(d-4)} T_{12}^\dagger + \\
& + \frac{(2(d-4)s_{12}^2 + (d-2)(d-6)s_{13}s_{23})}{2s_{12}^2 s_{13}^2 s_{23}^2(d-3)(d-4)} T_{21}^\dagger\,, \\
\mathcal{P}(B) \;=\; & \frac{1}{2s_{12}^2 s_{13} s_{23}(d-3)} T_{11}^\dagger - \frac{1}{2s_{12}^2 s_{13} s_{23}(d-3)} T_{22}^\dagger + \frac{1}{2s_{12}^2 s_{13} s_{23}(d-4)} T^\dagger - \\
& - \frac{(2s_{13}-(d-4)s_{23})}{2s_{12}^2 s_{13}^2 s_{23}(d-3)(d-4)} T_{12}^\dagger + \frac{(2s_{23}-(d-4)s_{13})}{2s_{12}^2 s_{13} s_{23}^2(d-3)(d-4)} T_{21}^\dagger\,.
\end{aligned} \tag{2.16}$$

The projectors for A_{21} and A_{22} are obtained by the permutations

$$\begin{aligned}
\mathcal{P}(A_{22}) &= \mathcal{P}(A_{11}) \left\{ s_{13} \leftrightarrow s_{23}, T_{11}^\dagger \leftrightarrow T_{22}^\dagger, T_{12}^\dagger \leftrightarrow T_{21}^\dagger, T^\dagger \to -T^\dagger \right\}, \\
\mathcal{P}(A_{21}) &= \mathcal{P}(A_{12}) \left\{ s_{13} \leftrightarrow s_{23}, T_{11}^\dagger \leftrightarrow T_{22}^\dagger, T_{12}^\dagger \leftrightarrow T_{21}^\dagger, T^\dagger \to -T^\dagger \right\}.
\end{aligned} \tag{2.17}$$

2.3 Reconstruction of the matrix elements

Starting from the tensor coefficients, the interference (or square) of amplitudes can be easily extracted in a variety of schemes. In the HV scheme, the external particles are treated in 4-dimensions and we find that,

$$\begin{aligned}
\sum_{\text{spins}} \langle \mathcal{M} | \mathcal{M} \rangle \;=\; & + \frac{s_{13}^3 s_{23}}{2} (2B - A_{11} + A_{12})(2B - A_{11} + A_{12})^\dagger + \\
& + \frac{s_{13} s_{23}^3}{2} (2B - A_{21} + A_{22})(2B - A_{21} + A_{22})^\dagger + \\
& + \frac{s_{13} s_{23} s_{12}^2}{2} \left(A_{11} A_{11}^\dagger + A_{22} A_{22}^\dagger \right).
\end{aligned} \tag{2.18}$$

Note that the tensor coefficients are in general matrices in colour space and the colour sum has still to be performed in evaluating, for example, $A_{11}A_{11}^\dagger$. In CDR, additional terms of order ϵ and ϵ^2 are straightforwardly generated. The resulting formula is, however, rather lengthy and we do not give it here.

3. The perturbative expansion of the tensor coefficients

The three processes that we wish to consider in detail are,

$$0 \to q\bar{q}gg\,, \tag{3.1}$$
$$0 \to q\bar{q}g\gamma\,, \tag{3.2}$$
$$0 \to q\bar{q}\gamma\gamma\,. \tag{3.3}$$

The amplitude for the process $\mathcal{P}$ (where $\mathcal{P} = gg,\ g\gamma$ and $\gamma\gamma$) has the perturbative expansion of the form,

$$|\mathcal{M}^{\mathcal{P}}\rangle = c^{\mathcal{P}}\left[|\mathcal{M}^{\mathcal{P},(0)}\rangle + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{\mathcal{P},(1)}\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2 |\mathcal{M}^{\mathcal{P},(2)}\rangle + \mathcal{O}(\alpha_s^3)\right], \tag{3.4}$$

where the overall coupling factor is given by

$$c^{\mathcal{P}} = 4\pi\,\alpha_s^q\,(\alpha e_q^2)^{1-q}\,, \tag{3.5}$$

with $q = 1, 1/2, 0$ for $\mathcal{P} = gg, g\gamma, \gamma\gamma$ respectively. $|\mathcal{M}^{\mathcal{P},(n)}\rangle$ denotes the n-loop contribution to the amplitude. Denoting the tensor coefficients as $X^{\mathcal{P}}$ (for $X = A_{IJ}$ and B), we find the perturbative expansions of the form

$$X_{IJ}^{\mathcal{P}} = c^{\mathcal{P}}\left[X_{IJ}^{\mathcal{P},(0)} + \left(\frac{\alpha_s}{2\pi}\right)X_{IJ}^{\mathcal{P},(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 X_{IJ}^{\mathcal{P},(2)} + \mathcal{O}(\alpha_s^3)\right], \tag{3.6}$$

where the dependence on the kinematic scales is implicit. The n-loop perturbative coefficients are vectors in colour space and can be further decomposed as,

$$\begin{aligned}
X^{gg,(n)} &= \sum_{i=1}^{3} \mathcal{C}^{gg,[i]}\, X^{gg,(n),[i]}\,,\\
X^{g\gamma,(n)} &= \mathcal{C}^{g\gamma,[1]}\, X^{g\gamma,(n),[1]}\,,\\
\mathcal{C}^{\gamma\gamma,[1]} &= \delta_{i_1 i_2}\, X^{\gamma\gamma,(n),[1]}\,,
\end{aligned} \tag{3.7}$$

where

$$\mathcal{C}^{gg,[1]} = (\boldsymbol{T}^{a_3}\boldsymbol{T}^{a_4})_{i_2 i_1}\,, \qquad \mathcal{C}^{gg,[2]} = (\boldsymbol{T}^{a_4}\boldsymbol{T}^{a_3})_{i_2 i_1}\,, \qquad \mathcal{C}^{gg,[3]} = \boldsymbol{\delta}^{a_3 a_4}\delta_{i_1 i_2}\,, \tag{3.8}$$

and

$$\mathcal{C}^{g\gamma,[1]} = (\boldsymbol{T}^{a_3})_{i_2 i_1}\,, \qquad \mathcal{C}^{\gamma\gamma,[1]} = \delta_{i_1 i_2}\,. \tag{3.9}$$

Index i denotes a colour label in the fundamental representation while index a is in the adjoint. Note that the normalisation of the colour matrices is given by $Tr(\boldsymbol{T}^{a_1}\boldsymbol{T}^{a_2}) = \frac{1}{2}\boldsymbol{\delta}^{a_1 a_2}$.

Symmetry under interchange of the two gluons gives the additional relations,

$$\begin{aligned}
A_{IJ}^{gg,(n),[2]}(s_{13}, s_{23}) &= -A_{JI}^{gg,(n),[1]}(s_{23}, s_{13})\,,\\
A_{IJ}^{gg,(n),[3]}(s_{13}, s_{23}) &= -A_{JI}^{gg,(n),[3]}(s_{23}, s_{13})\,,\\
B^{gg,(n),[2]}(s_{13}, s_{23}) &= B^{gg,(n),[1]}(s_{23}, s_{13})\,,\\
B^{gg,(n),[3]}(s_{13}, s_{23}) &= B^{gg,(n),[3]}(s_{23}, s_{13})\,.
\end{aligned} \tag{3.10}$$

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For the processes involving one or more photons,

$$\begin{aligned}
A_{IJ}^{g\gamma,(n),[1]}(s_{13},s_{23}) &= -A_{JI}^{g\gamma,(n),[1]}(s_{23},s_{13})\,,\\
A_{IJ}^{\gamma\gamma,(n),[1]}(s_{13},s_{23}) &= -A_{JI}^{\gamma\gamma,(n),[1]}(s_{23},s_{13})\,,\\
B^{g\gamma,(n),[1]}(s_{13},s_{23}) &= B^{g\gamma,(n),[1]}(s_{23},s_{13})\,,\\
B^{\gamma\gamma,(n),[1]}(s_{13},s_{23}) &= B^{\gamma\gamma,(n),[1]}(s_{23},s_{13})\,.
\end{aligned} \tag{3.11}$$

As in previous work, we use QGRAF [54] to generate the Feynman diagrams and then use MAPLE [55] and FORM [56] to manipulate the Dirac structures and traceology and then reduce the scalar loop integrals to master integrals.

3.1 Tree level results

At tree level, none of the T_{IJ} structures are present so that,

$$A_{IJ}^{\mathcal{P},(0),[i]}(s_{13},s_{23}) = 0\,, \tag{3.12}$$

while only the first two colour structures contribute to $B^{gg,(0)}$,

$$\begin{aligned}
B^{gg,(0),[1]}(s_{13},s_{23}) &= -\frac{2}{s_{12}s_{13}}\,,\\
B^{gg,(0),[2]}(s_{13},s_{23}) &= -\frac{2}{s_{12}s_{23}}\,,\\
B^{gg,(0),[3]}(s_{13},s_{23}) &= 0\,.
\end{aligned} \tag{3.13}$$

For the other processes,

$$\begin{aligned}
B^{g\gamma,(0),[1]}(s_{13},s_{23}) &= B^{\gamma\gamma,(0),[1]}(s_{13},s_{23})\\
&= B^{gg,(0),[1]}(s_{13},s_{23}) + B^{gg,(0),[2]}(s_{13},s_{23})\\
&= \frac{2}{s_{13}s_{23}}\,.
\end{aligned} \tag{3.14}$$

3.2 One-loop results

At one-loop, all of the tensor and colour structures contribute. For the two-gluon process, we find the following (unrenormalised) structure,

$$\begin{aligned}
A_{11}^{gg,(1),[1]}(s_{13},s_{23}) &= C_F\,f_1(s_{13},s_{23}) + C_A\,g_1(s_{13},s_{23}) + (2N_F - C_A(d-2))\,h(s_{12})\,,\\
A_{11}^{gg,(1),[2]}(s_{13},s_{23}) &= -A_{11}^{gg,(1),[1]}(s_{23},s_{13})\,,\\
A_{11}^{gg,(1),[3]}(s_{13},s_{23}) &= 0\,,\\
A_{12}^{gg,(1),[1]}(s_{13},s_{23}) &= C_F\,f_2(s_{13},s_{23}) + C_A\,g_2(s_{13},s_{23}) + (2N_F - C_A(d-2))\,h(s_{12})\,,\\
A_{12}^{gg,(1),[2]}(s_{13},s_{23}) &= C_F\,f_3(s_{13},s_{23}) + C_A\,g_3(s_{13},s_{23}) - (2N_F - C_A(d-2))\,h(s_{12})\,,\\
A_{12}^{gg,(1),[3]}(s_{13},s_{23}) &= h_1(s_{13},s_{23})\,,\\
B^{gg,(1),[1]}(s_{13},s_{23}) &= C_F\,f_4(s_{13},s_{23}) + C_A\,g_4(s_{13},s_{23})\,,\\
B^{gg,(1),[2]}(s_{13},s_{23}) &= B^{gg,(1),[1]}(s_{23},s_{13})\,,\\
B^{gg,(1),[3]}(s_{13},s_{23}) &= h_2(s_{13},s_{23})\,,
\end{aligned} \tag{3.15}$$

while for the gluon-photon and photon-photon processes we have,

$$\begin{aligned}
A_{11}^{g\gamma,(1),[1]}(s_{13},s_{23}) &= C_F\left(f_1(s_{13},s_{23}) - f_1(s_{23},s_{13})\right) + \\
&\quad + \frac{C_A}{2}\left(g_1(s_{13},s_{23}) - g_1(s_{23},s_{13}) - 4h_3(s_{13},s_{23})\right), \\
A_{12}^{g\gamma,(1),[1]}(s_{13},s_{23}) &= C_F\left(f_2(s_{13},s_{23}) + f_3(s_{13},s_{23})\right) + \\
&\quad + \frac{C_A}{2}\left(g_2(s_{13},s_{23}) + g_3(s_{13},s_{23}) - 2h_1(s_{13},s_{23})\right), \\
B^{g\gamma,(1),[1]}(s_{13},s_{23}) &= C_F\left(f_4(s_{13},s_{23}) + f_4(s_{23},s_{13})\right) + \\
&\quad + \frac{C_A}{2}\left(g_4(s_{13},s_{23}) + g_4(s_{23},s_{13}) - 2h_2(s_{13},s_{23}) - 2h_3(s_{13},s_{23})\right),
\end{aligned} \tag{3.16}$$

and,

$$\begin{aligned}
A_{11}^{\gamma\gamma,(1),[1]}(s_{13},s_{23}) &= C_F\left(f_1(s_{13},s_{23}) - f_1(s_{23},s_{13})\right), \\
A_{12}^{\gamma\gamma,(1),[1]}(s_{13},s_{23}) &= C_F\left(f_2(s_{13},s_{23}) + f_3(s_{13},s_{23})\right), \\
B^{\gamma\gamma,(1),[1]}(s_{13},s_{23}) &= C_F\left(f_4(s_{13},s_{23}) + f_4(s_{23},s_{13})\right).
\end{aligned} \tag{3.17}$$

The functions f_i, g_i, h_i and h can be written in terms of the one-loop box integral in $d = 6-2\epsilon$ dimensions, $\mathrm{Box}^6(s_{ij},s_{ik})$, and the one-loop bubble, $\mathrm{Bub}(s_{ij})$. For all orders in $\epsilon = (4-d)/2$, we find that,

$$h(s_{12}) = \frac{2(d-4)}{s_{12}^2(d-1)(d-2)}\,\mathrm{Bub}(s_{12}), \tag{3.18}$$

while,

$$\begin{aligned}
f_1(s_{13},s_{23}) &= \frac{(d-2)(d-4)}{s_{13}}\,\mathrm{Box}^6(s_{12},s_{23}) - \frac{2(d-4)}{s_{13}s_{23}}\,\mathrm{Bub}(s_{23}) - \frac{2(d-4)}{s_{12}s_{13}}\,\mathrm{Bub}(s_{12}), \\
f_2(s_{13},s_{23}) &= \frac{s_{23}^2 d(d-2) - s_{13}s_{23}(d-2)(d-14) - 2s_{13}^2(d-10)}{s_{12}s_{13}^2}\,\mathrm{Box}^6(s_{12},s_{23}) + \\
&\quad + \frac{2(d-12)s_{13} - 2s_{23}d}{s_{12}s_{13}^2}\,\mathrm{Bub}(s_{23}) + \frac{2s_{23}d - 4(d-8)s_{13}}{s_{12}s_{13}^2}\,\mathrm{Bub}(s_{12}), \\
f_3(s_{13},s_{23}) &= \frac{((d-4)(d-10)s_{23}^2 + (3d^2-36d+92)s_{13}s_{23} - 2(d-2)s_{13}^2}{s_{12}s_{13}s_{23}}\,\mathrm{Box}^6(s_{12},s_{13}) + \\
&\quad + \frac{4s_{13} - 4(d-7)s_{23}}{s_{12}s_{13}s_{23}}\,\mathrm{Bub}(s_{13}) + \frac{2(d-10)s_{23} - 4s_{13}}{s_{12}s_{13}s_{23}}\,\mathrm{Bub}(s_{12}), \\
f_4(s_{13},s_{23}) &= + \frac{4(d-3)s_{13}^2 - (d^2-15d+38)s_{23}s_{13} + (d-2)s_{23}^2}{s_{12}s_{13}s_{23}}\,\mathrm{Box}^6(s_{12},s_{23}) + \\
&\quad + \frac{2(d-7)s_{13} - 2s_{23}}{s_{12}s_{13}s_{23}}\,\mathrm{Bub}(s_{23}) + \frac{2(d-4)s_{23} + 8(d-3)s_{13}}{(d-4)s_{12}s_{13}s_{23}}\,\mathrm{Bub}(s_{12}), \\
g_1(s_{13},s_{23}) &= -f_1(s_{13},s_{23}), \\
g_2(s_{13},s_{23}) &= -\frac{s_{23}(d-2)(4s_{13}+ds_{23})}{s_{12}s_{13}^2}\,\mathrm{Box}^6(s_{12},s_{23}) + \frac{2s_{23}d + 4s_{13}}{s_{12}s_{13}^2}\,\mathrm{Bub}(s_{23}) + \\
&\quad + \frac{2(d-6)s_{13} - 2ds_{23}}{s_{12}s_{13}^2}\,\mathrm{Bub}(s_{12}),
\end{aligned}$$

$$
\begin{aligned}
g_3(s_{13},s_{23}) &= \frac{(d-2)(2s_{13}-(d-6)s_{23})}{s_{12}s_{23}}\,\mathrm{Box}^6(s_{12},s_{13}) + \frac{4}{s_{12}s_{23}}\,\mathrm{Bub}(s_{12}) + \\
&\quad + \frac{2(d-4)s_{23}-4s_{13}}{s_{12}s_{13}s_{23}}\,\mathrm{Bub}(s_{13})\,, \\
g_4(s_{13},s_{23}) &= \frac{(d^2-16d+36)s_{13}-2(d-2)s_{23}}{2s_{12}s_{13}}\,\mathrm{Box}^6(s_{12},s_{23}) - \\
&\quad - \frac{(d^2-16d+40)s_{13}-2(d-4)s_{23}}{(d-4)s_{12}s_{13}s_{23}}\,\mathrm{Bub}(s_{23}) - \frac{2}{s_{12}s_{13}}\,\mathrm{Bub}(s_{12})\,, \\
h_1(s_{13},s_{23}) &= -\frac{(d-10)((d-2)s_{23}+2s_{13})s}{2s_{12}s_{13}}\,\mathrm{Box}^6(s_{12},s_{23}) + \\
&\quad + \frac{(d-4)(d-10)(2s_{13}+s_{23})}{2s_{12}s_{13}}\,\mathrm{Box}^6(s_{12},s_{13}) - \\
&\quad - \frac{(d-10)(2s_{13}-(d-4)s_{23})}{2s_{12}s_{13}}\,\mathrm{Box}^6(s_{13},s_{23})\,, \\
h_2(s_{13},s_{23}) &= \frac{(8(d-3)s_{13}-(d-4)(d-10)s_{23})}{4s_{12}s_{23}}\left(\mathrm{Box}^6(s_{12},s_{23}) + \mathrm{Box}^6(s_{13},s_{23})\right) + \\
&\quad + \frac{(8(d-3)s_{23}-(d-4)(d-10)s_{13})}{4s_{12}s_{13}}\left(\mathrm{Box}^6(s_{12},s_{13}) + \mathrm{Box}^6(s_{13},s_{23})\right) + \\
&\quad + \frac{4+3(d-4)^2}{4s_{12}}\,\mathrm{Box}^6(s_{13},s_{23}) - \\
&\quad - \frac{4(d-3)}{(d-4)}\left(\frac{1}{s_{12}s_{13}}\,\mathrm{Bub}(s_{23}) + \frac{1}{s_{12}s_{23}}\,\mathrm{Bub}(s_{13}) + \frac{1}{s_{23}s_{13}}\,\mathrm{Bub}(s_{12})\right), \\
h_3(s_{13},s_{23}) &= \frac{1}{4}f_1(s_{12},s_{23})\,.
\end{aligned}
\tag{3.19}
$$

3.3 Ultraviolet renormalisation

The renormalisation of the matrix element is carried out by replacing the bare coupling α_0 with the renormalised coupling $\alpha_s \equiv \alpha_s(\mu^2)$, evaluated at the renormalisation scale μ^2 according to eq. (2.4).

We denote the i-loop contribution to the unrenormalised tensor coefficients for process $\mathcal{P}$ by $A_{IJ}^{\mathcal{P},(i),\mathrm{un}}$ and $B^{\mathcal{P},(i),\mathrm{un}}$, using the same normalization as for the decomposition of the tensor coefficients (3.4); the dependence on (s_{13},s_{23}) is always understood implicitly. The renormalised coefficients are then obtained as

$$
\begin{aligned}
A_{IJ}^{\mathcal{P},(0)} &= 0\,, \\
A_{IJ}^{\mathcal{P},(1)} &= S_\epsilon^{-1} A_{IJ}^{\mathcal{P},(1),\mathrm{un}}\,, \\
A_{IJ}^{\mathcal{P},(2)} &= S_\epsilon^{-2} A_{IJ}^{\mathcal{P},(2),\mathrm{un}} - \frac{(q+1)\beta_0}{\epsilon} S_\epsilon^{-1} A_{IJ}^{\mathcal{P},(1),\mathrm{un}}\,,
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
B^{\mathcal{P},(0)} &= B^{\mathcal{P},(0),\mathrm{un}}\,, \\
B^{\mathcal{P},(1)} &= S_\epsilon^{-1} B^{\mathcal{P},(1),\mathrm{un}} - \frac{q\beta_0}{\epsilon} B^{\mathcal{P},(0),\mathrm{un}}\,, \\
B^{\mathcal{P},(2)} &= S_\epsilon^{-2} B^{\mathcal{P},(2),\mathrm{un}} - \frac{(q+1)\beta_0}{\epsilon} S_\epsilon^{-1} B^{\mathcal{P},(1),\mathrm{un}} - \frac{q}{2}\left(\frac{\beta_1}{\epsilon} - \frac{(q+1)\beta_0^2}{\epsilon^2}\right) B^{\mathcal{P},(0),\mathrm{un}}\,,
\end{aligned}
\tag{3.21}
$$

where q is the number of powers of the strong coupling for process $\mathcal{P}$ as described in eq. (3.5).

3.4 Infrared behaviour of the tensor coefficients

After performing ultraviolet renormalisation, the amplitudes still contain singularities which are of infrared origin and will be analytically cancelled by those occurring in radiative processes of the same order. Catani [49] has shown how to organize the infrared pole structure of the one- and two-loop contributions renormalised in the $\overline{\mathrm{MS}}$ scheme in terms of the tree and renormalised one-loop amplitudes. The same procedure applies to the tensor coefficients.

In particular, the infrared behaviour of the one-loop coefficients is given by

$$\begin{aligned} A_{IJ}^{\mathcal{P},(1)} &= A_{IJ}^{\mathcal{P},(1),\mathrm{finite}}, \\ B^{\mathcal{P},(1)} &= \boldsymbol{I}^{\mathcal{P},(1)}(\epsilon)B^{\mathcal{P},(0)} + B^{\mathcal{P},(1),\mathrm{finite}}, \end{aligned} \tag{3.22}$$

while the two-loop singularity structure is

$$\begin{aligned} A_{IJ}^{\mathcal{P},(2)} &= \boldsymbol{I}^{\mathcal{P},(1)}(\epsilon)A_{IJ}^{\mathcal{P},(1)} + A_{IJ}^{\mathcal{P},(2),\mathrm{finite}}, \\ B^{\mathcal{P},(2)} &= \boldsymbol{I}^{\mathcal{P},(2)}(\epsilon)B^{\mathcal{P},(0)} + \boldsymbol{I}^{\mathcal{P},(1)}(\epsilon)B^{(1)} + B^{(2),\mathrm{finite}}, \end{aligned} \tag{3.23}$$

where $\boldsymbol{I}^{\mathcal{P},(2)}(\epsilon)$ is given by,

$$\begin{aligned} \boldsymbol{I}^{\mathcal{P},(2)}(\epsilon) = &-\frac{1}{2}\boldsymbol{I}^{\mathcal{P},(1)}(\epsilon)\boldsymbol{I}^{\mathcal{P},(1)}(\epsilon) - \frac{\beta_0}{\epsilon}\boldsymbol{I}^{\mathcal{P},(1)}(\epsilon) + e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\boldsymbol{I}^{\mathcal{P},(1)}(2\epsilon) \\ &+\boldsymbol{H}^{\mathcal{P},(2)}(\epsilon) \end{aligned} \tag{3.24}$$

and the constant K is

$$K = \left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A - \frac{10}{9}T_R N_F\,. \tag{3.25}$$

In QCD the $\boldsymbol{I}^{\mathcal{P},(1)}(\epsilon)$ operator is given by [49]

$$\boldsymbol{I}^{\mathcal{P},(1)}(\epsilon) = \frac{1}{2}\frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\sum_i \nu_i^{sing}(\epsilon)\sum_{j\neq i}\mathbf{T}_i\cdot\mathbf{T}_j\left(-\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} \tag{3.26}$$

where the sum runs over the pairs of external coloured particles involved in process $\mathcal{P}$ and the singular function is

$$\nu_i^{sing}(\epsilon) = \frac{1}{\epsilon^2}+\gamma_i\frac{1}{\epsilon}\,, \qquad \gamma_{q,\bar{q}} = \frac{3}{2}\,, \qquad \gamma_g = \frac{\beta_0}{C_A}\,. \tag{3.27}$$

The colour charge algebra is defined as

$$\mathbf{T}_i\cdot\mathbf{T}_j = \begin{cases} \mathbf{T}_j\cdot\mathbf{T}_i & \text{if } i\neq j\,, \\ \mathbf{T}_i^2 = C_i & \text{otherwise}\,, \end{cases}$$

with the Casimir $C_i = C_F$ ($C_i = C_A$) if parton i is a quark (gluon) and where the colour charge $\mathbf{T}_i$ is t^a_{cb} ($-t^a_{cb}$) for a quark (anti-quark) and if_{cab} for a gluon.

Then, given the colour basis for the two-quark two-gluon process presented in eq. (3.7), we have the following infrared-singularity operator

$$\boldsymbol{I}^{gg,(1)}(\epsilon) = \frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)} \begin{pmatrix} \boldsymbol{A}(\epsilon, s_{12}, s_{13}, s_{23}) & 0 & \boldsymbol{B}(\epsilon, s_{23}, s_{13}, s_{12}) \\ 0 & \boldsymbol{A}(\epsilon, s_{12}, s_{23}, s_{13}) & \boldsymbol{B}(\epsilon, s_{13}, s_{23}, s_{12}) \\ \boldsymbol{B}(\epsilon, s_{12}, s_{13}, s_{23}) & \boldsymbol{B}(\epsilon, s_{12}, s_{23}, s_{13}) & \boldsymbol{C}(\epsilon, s_{12}) \end{pmatrix} \tag{3.28}$$

where

$$\begin{aligned} \boldsymbol{A}(\epsilon, s_{12}, s_{23}, s_{13}) &= -\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right)\left[N\left(-\frac{\mu^2}{s_{13}}\right)^{\epsilon} + C_F\left(-\frac{\mu^2}{s_{12}}\right)^{\epsilon}\right] + \\ &\quad + N\left(\frac{\beta_0}{2N\epsilon} - \frac{3}{4\epsilon}\right)\left[\left(-\frac{\mu^2}{s_{12}}\right)^{\epsilon} + \left(-\frac{\mu^2}{s_{13}}\right)^{\epsilon}\right], \qquad (3.29) \\ \boldsymbol{B}(\epsilon, s_{12}, s_{23}, s_{13}) &= \left(\frac{1}{\epsilon^2} + \frac{3}{4\epsilon} + \frac{\beta_0}{2N\epsilon}\right)\left[\left(-\frac{\mu^2}{s_{23}}\right)^{\epsilon} - \left(-\frac{\mu^2}{s_{12}}\right)^{\epsilon}\right], \qquad (3.30) \\ \boldsymbol{C}(\epsilon, s_{12}) &= -\left[\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right)C_F + \left(\frac{1}{\epsilon^2} + \frac{\beta_0}{N\epsilon}\right)N\right]\left(-\frac{\mu^2}{s_{12}}\right)^{\epsilon}. \qquad (3.31) \end{aligned}$$

Similarly, for the other processes we have

$$\begin{aligned} \boldsymbol{I}^{g\gamma,(1)}(\epsilon) &= \frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)}\left\{\frac{1}{N}\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right)\left(-\frac{\mu^2}{s_{12}}\right)^{\epsilon} - \right. \\ &\quad \left. - N\left(\frac{1}{\epsilon^2} + \frac{3}{4\epsilon} + \frac{\beta_0}{2N\epsilon}\right)\left[\left(-\frac{\mu^2}{s_{23}}\right)^{\epsilon} + \left(-\frac{\mu^2}{s_{13}}\right)^{\epsilon}\right]\right\} \qquad (3.32) \end{aligned}$$

and

$$\boldsymbol{I}^{\gamma\gamma,(1)}(\epsilon) = -C_F\frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right)\left(-\frac{\mu^2}{s_{12}}\right)^{\epsilon}. \tag{3.33}$$

In eq. (3.23) the function $\boldsymbol{H}^{\mathcal{P},(2)}$ contains poles of $\mathcal{O}(1/\epsilon)$ and is process and renormalisation scheme dependent. The specific structure of this term was not given in ref. [49]. However, there it was pointed out that in the case of a QCD process that involves a $q\bar{q}$ pair, the function $\boldsymbol{H}^{\mathcal{P},(2)}$ is related to the quark electromagnetic form factor. Now, we know that the form factor is not merely related, but is the backbone of the singular structure of multi-loop QCD amplitudes [57].

It was only after the direct Feynman diagram evaluation of two-loop matrix elements [58, 59, 60, 1, 61, 2], that the colour uncorrelated part of the $\boldsymbol{H}^{\mathcal{P},(2)}$ function for all $2 \to 2$ partonic processes was fully identified. More precisely, it was found that each external coloured leg in the partonic process contributes independently to the matrix element given by

$$\langle \mathcal{M}_0 | \boldsymbol{H}^{\mathcal{P},(2)} | \mathcal{M}_0 \rangle = \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)} H^{\mathcal{P},(2)} \langle \mathcal{M}_0 | \mathcal{M}_0 \rangle \tag{3.34}$$

with

$$H^{\mathcal{P},(2)} = n_q H_q^{(2)} + n_g H_g^{(2)} \tag{3.35}$$

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where n_g is the number of external gluons, n_q is the number of external quarks and anti-quarks for the process $\mathcal{P}$ and

$$\begin{aligned} H_q^{(2)} = H_{\bar{q}}^{(2)} &= \left(\frac{\pi^2}{2} - 6\,\zeta_3 - \frac{3}{8}\right) C_F^2 + \left(\frac{13}{2}\zeta_3 + \frac{245}{216} - \frac{23}{48}\pi^2\right) C_A C_F + \\ &\quad + \left(-\frac{25}{54} + \frac{\pi^2}{12}\right) T_R N_F C_F \,, \end{aligned} \tag{3.36}$$

$$\begin{aligned} H_g^{(2)} &= \frac{20}{27} T_R^2 N_F^2 + T_R C_F N_F - \left(\frac{\pi^2}{36} + \frac{58}{27}\right) T_R N_F C_A + \\ &\quad + \left(\frac{\zeta_3}{2} + \frac{5}{12} + \frac{11}{144}\pi^2\right) C_A^2 \,. \end{aligned} \tag{3.37}$$

Already in ref. [49], Catani pointed out the possibility of $\boldsymbol{H}^{\mathcal{P},(2)}$ having additional non-trivial colour correlations which could be explored by calculating the hard scattering amplitude, rather than the interference of two-loop and tree amplitudes.[1] Such colour correlations were found for the four gluon interaction, which is the only partonic process (involving four coloured particles) for which the two-loop helicity amplitudes [30] exist. According to ref. [30], the colour correlations form a non-trivial matrix in colour space and is the same for all helicities.

For the processes we study, we find that the colour correlations are indeed helicity independent and are given by

$$\boldsymbol{H}^{\mathcal{P},(2)}(\epsilon) = \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)} \left(-\frac{\mu^2}{s_{12}}\right)^{2\epsilon} \left(H^{\mathcal{P},(2)}\ \mathbf{1} + \hat{\mathbf{H}}^{\mathcal{P},(2)}\right) \tag{3.38}$$

with $H^{\mathcal{P},(2)}$ as given in eq. (3.35) and

$$\hat{\mathbf{H}}^{gg,(2)} = -4\ i\ f_{abc}\ \boldsymbol{T}^{a[1]}\boldsymbol{T}^{b[2]}\boldsymbol{T}^{c[3]}\ \ln\left(\frac{-s_{12}}{-s_{23}}\right) \ln\left(\frac{-s_{23}}{-s_{13}}\right) \ln\left(\frac{-s_{13}}{-s_{12}}\right), \tag{3.39}$$

$$\hat{\mathbf{H}}^{g\gamma,(2)} = 0\,, \tag{3.40}$$

$$\hat{\mathbf{H}}^{\gamma\gamma,(2)} = 0\,. \tag{3.41}$$

These results are in complete agreement with the results found in ref. [30] and show that the colour correlations are also independent of the external particles. This also agrees with the analysis of ref. [57] where it was shown that the contributions to $\hat{\mathbf{H}}^{\mathcal{P},(2)}$ are of soft nature and directly related to soft anomalous dimension matrices obtained by studying the evolution of colour exchange in QCD hard scattering (see [62] and references therein). It is worth mentioning that these type of colour correlations between three different partons also appeared in the analysis of higher-order contributions to the soft-gluon current in ref. [63].

Note that the matrix $f_{abc}\,\boldsymbol{T}^{a[1]}\boldsymbol{T}^{b[2]}\boldsymbol{T}^{c[3]}$ collects all colour weights that result from the analysis of triple gluon vertices connecting any three different external partons, projected into the colour basis given in eq. (3.7), so that for the two quark-two gluon process,

$$i\ f_{abc}\,\boldsymbol{T}^{a[1]}\boldsymbol{T}^{b[2]}\boldsymbol{T}^{c[3]} = -\frac{1}{8}\begin{pmatrix} 2 & 2 & 4N \\ -2 & -2 & -4N \\ -N & N & 0 \end{pmatrix}. \tag{3.42}$$

[1] The sum over colours causes the colour correlations to vanish

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Given the tree level results presented in section 3.1 and the divergent structure presented in eq. (3.23), we can see that in fact the third column of this matrix is not needed at $\mathcal{O}(1/\epsilon)$, since the $\boldsymbol{\delta}^{a_3a_4}\delta_{i_1i_2}$ colour structure is not present at tree level.

4. Helicity amplitudes

The helicity amplitudes $|\mathcal{M}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4}$ can be obtained from the general d-dimensional tensor of eq. (2.10) using standard four-dimensional helicity techniques corresponding to working in the 't Hooft-Veltman scheme. Conventionally, the two helicity states of a four-dimensional light-like spinor $\psi(p)$ are denoted by,

$$\psi_\pm(p) = \frac{1}{2}(1 \pm \gamma_5)\psi(p)\,, \tag{4.1}$$

with the further notation,

$$|p\pm\rangle = \psi_\pm(p)\,, \qquad \langle p\pm| = \overline{\psi_\pm(p)}\,. \tag{4.2}$$

Particles may thus be crossed to the initial state by reversing the sign of the helicity. The basic quantity is the spinor product,

$$\langle pq\rangle = \langle p - |q+\rangle\,, \qquad [pq] = [p + |q-]\,, \tag{4.3}$$

such that

$$\langle pq\rangle[qp] = 2p.q\,. \tag{4.4}$$

In this notation, the polarization vector of the out-going light-like gauge boson of momentum p is defined with respect to a reference momentum k as,

$$\epsilon^\pm(p,k) = \pm\frac{\langle p \pm |\gamma_\mu|k\pm\rangle}{\sqrt{2}\langle k \mp |p\pm\rangle}\,. \tag{4.5}$$

Fixing the helicity of the (final-state) quark to be positive, $\lambda_1 = +$ we find that the helicity amplitudes, upto a helicity dependent phase, are given by

$$\begin{aligned}
|\mathcal{M}\rangle_{+-++} &\propto \frac{\sqrt{s_{12}^2 s_{13} s_{23}}}{2} A_{11}(s_{13}, s_{23})\,,\\
|\mathcal{M}\rangle_{+-+-} &\propto \frac{\sqrt{s_{13}^3 s_{23}}}{2}\left(2B(s_{13}, s_{23}) + A_{12}(s_{13}, s_{23}) - A_{11}(s_{13}, s_{23})\right),\\
|\mathcal{M}\rangle_{+--+} &\propto \frac{\sqrt{s_{13} s_{23}^3}}{2}\left(2B(s_{13}, s_{23}) + A_{22}(s_{13}, s_{23}) - A_{21}(s_{13}, s_{23})\right),\\
|\mathcal{M}\rangle_{+---} &\propto -\frac{\sqrt{s_{12}^2 s_{13} s_{23}}}{2} A_{22}(s_{13}, s_{23})\,.
\end{aligned} \tag{4.6}$$

By inspection of eqs. (3.10) and (3.11) we see that, as expected, there are two independent helicity amplitudes and that the other two helicity amplitudes can be obtained by Bose symmetry and,

$$\begin{aligned}
|\mathcal{M}\rangle_{+--+} &= |\mathcal{M}\rangle_{+-+-}(s_{13} \leftrightarrow s_{23})\,,\\
|\mathcal{M}\rangle_{+---} &= |\mathcal{M}\rangle_{+-++}(s_{13} \leftrightarrow s_{23})\,.
\end{aligned} \tag{4.7}$$

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Helicity amplitudes where the quark helicity is negative are obtained through parity,

$$|\mathcal{M}\rangle_{-+-\lambda_3-\lambda_4} = \left(|\mathcal{M}\rangle_{+-\lambda_3\lambda_4}\right)^* . \tag{4.8}$$

By inspection, we see that "squaring" and summing over the helicities generates the interference of amplitudes given in eq. (2.18). Thus, the helicity amplitudes can be obtained directly from a Feynman diagram calculation by projecting out the individual tensor coefficients and taking the appropriate linear combination.

Just as for the individual tensor coefficients, the helicity amplitudes for each process $\mathcal{P}$ can be perturbatively decomposed as,

$$\begin{aligned}|\mathcal{M}^{\mathcal{P}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} = c^{\mathcal{P}}\Bigg[&|\mathcal{M}^{\mathcal{P},(0)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{\mathcal{P},(1)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} + \\ &+ \left(\frac{\alpha_s}{2\pi}\right)^2|\mathcal{M}^{\mathcal{P},(2)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} + \mathcal{O}(\alpha_s^3)\Bigg],\end{aligned} \tag{4.9}$$

where the overall coupling factor is given by eq. (3.5). As in eqs. (3.20) and (3.21), the renormalised helicity amplitudes are obtained from the unrenormalised amplitudes by

$$\begin{aligned}|\mathcal{M}^{\mathcal{P},(0)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} &= |\mathcal{M}^{\mathcal{P},(0),\mathrm{un}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4},\\ |\mathcal{M}^{\mathcal{P},(1)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} &= S_\epsilon^{-1}|\mathcal{M}^{\mathcal{P},(1),\mathrm{un}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} - \frac{q\beta_0}{\epsilon}|\mathcal{M}^{\mathcal{P},(0),\mathrm{un}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4},\\ |\mathcal{M}^{\mathcal{P},(2)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} &= S_\epsilon^{-2}|\mathcal{M}^{\mathcal{P},(2),\mathrm{un}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} - \frac{(q+1)\beta_0}{\epsilon}S_\epsilon^{-1}|\mathcal{M}^{\mathcal{P},(1),\mathrm{un}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} - \\ &\quad -\frac{q}{2}\left(\frac{\beta_1}{\epsilon} - \frac{(q+1)\beta_0^2}{\epsilon^2}\right)|\mathcal{M}^{\mathcal{P},(0),\mathrm{un}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4},\end{aligned} \tag{4.10}$$

while the infrared singularity structure is given by,

$$|\mathcal{M}^{\mathcal{P},(1)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} = \boldsymbol{I}^{\mathcal{P},(1)}(\epsilon)|\mathcal{M}^{\mathcal{P},(0)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} + |\mathcal{M}^{\mathcal{P},(1),\mathrm{finite}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4}, \tag{4.11}$$

$$\begin{aligned}|\mathcal{M}^{\mathcal{P},(2)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} &= \boldsymbol{I}^{\mathcal{P},(2)}(\epsilon)|\mathcal{M}^{\mathcal{P},(0)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} + \boldsymbol{I}^{\mathcal{P},(1)}(\epsilon)|\mathcal{M}^{\mathcal{P},(1)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} + \\ &\quad +|\mathcal{M}^{\mathcal{P},(2),\mathrm{finite}}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4},\end{aligned} \tag{4.12}$$

where the operators $\boldsymbol{I}^{\mathcal{P},(i)}$ are the process dependent matrices in colour space given in section 3.4.

Similarly, the n-loop helicity amplitudes are vectors in colour space and can be further decomposed as in eq. (3.7)

$$\begin{aligned}|\mathcal{M}^{gg,(n)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} &= \sum_{i=1}^{3}\mathcal{C}^{gg,[i]}\,|\mathcal{M}^{gg,(n),[i]}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4},\\ |\mathcal{M}^{g\gamma,(n)}\rangle &= \mathcal{C}^{g\gamma,[1]}\,|\mathcal{M}^{g\gamma,(n),[1]}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4},\\ |\mathcal{M}^{\gamma\gamma,(n)}\rangle &= \mathcal{C}^{\gamma\gamma,[1]}\,|\mathcal{M}^{\gamma\gamma,(n),[1]}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4}.\end{aligned} \tag{4.13}$$

4.1 Helicity amplitudes for physical processes

The physically relevant scattering amplitudes we wish to describe are the quark-antiquark annihilation processes,

$$q(p_2,+)+\bar{q}(p_1,-)\to b_3(p_3,+)+b_4(p_4,+)\,, \tag{4.14}$$
$$q(p_2,+)+\bar{q}(p_1,-)\to b_3(p_3,+)+b_4(p_4,-)\,, \tag{4.15}$$
$$q(p_2,+)+\bar{q}(p_1,-)\to b_3(p_3,-)+b_4(p_4,+)\,, \tag{4.16}$$
$$q(p_2,+)+\bar{q}(p_1,-)\to b_3(p_3,-)+b_4(p_4,-)\,, \tag{4.17}$$

and the quark-boson scattering processes,

$$q(p_2,+)+b_3(p_1,+)\to q(p_3,+)+b_4(p_4,+)\,, \tag{4.18}$$
$$q(p_2,+)+b_3(p_1,+)\to q(p_3,+)+b_4(p_4,-)\,, \tag{4.19}$$
$$q(p_2,+)+b_3(p_1,-)\to q(p_3,+)+b_4(p_4,+)\,, \tag{4.20}$$
$$q(p_2,+)+b_3(p_1,-)\to q(p_3,+)+b_4(p_4,-)\,, \tag{4.21}$$

$$q(p_2,+)+b_4(p_1,+)\to b_3(p_3,+)+q(p_4,+)\,, \tag{4.22}$$
$$q(p_2,+)+b_4(p_1,+)\to b_3(p_3,-)+q(p_4,+)\,, \tag{4.23}$$
$$q(p_2,+)+b_4(p_1,-)\to b_3(p_3,+)+q(p_4,+)\,, \tag{4.24}$$
$$q(p_2,+)+b_4(p_1,-)\to b_3(p_3,-)+q(p_4,+)\,. \tag{4.25}$$

where the bosons b_3 or b_4 are gluons or photons. Processes where the initial state quark has negative helicity are obtained by a parity transformation, while processes with two bosons in the initial state are given by time reversal. There are additional symmetries when the gauge bosons are identical and further symmetries between the various colour components for the two gluon process.

For convenience, we denote the processes of eqs. (4.14)–(4.17) as the s-channel, the processes of eqs. (4.18)–(4.21) as the u-channel and the processes of eqs. (4.22)–(4.25) as the t-channel. s, t and u are the usual Mandelstam variables, $s=s_{12}>0$, $t=s_{23}<0$, $u=s_{13}<0$ and $s+t+u=0$.

To present the helicity amplitudes for the various processes, it is convenient to organise the amplitude where particles 1 and 2 are in the initial state in terms of a spinor factor $\mathcal{S}$ and the colour factor $\mathcal{C}$ so that,

$$|\mathcal{M}_c^{\mathcal{P},(n)}\rangle_{\lambda_2\lambda_1\lambda_3\lambda_4}=\sum_i\;\mathcal{S}^{\mathcal{P},[i]}_{c\lambda_2\lambda_1\lambda_3\lambda_4}\times\mathcal{C}_c^{\mathcal{P},[i]}\times|\widehat{\mathcal{M}}_c^{\mathcal{P},(n),[i]}\rangle_{\lambda_2\lambda_1\lambda_3\lambda_4}\,, \tag{4.26}$$

where c $(=s,\ t,\ u)$ denotes the channel and the sum runs over the number of colour structures: three for the two-gluon processes and one otherwise.

Explicitly, we have

$$\begin{aligned}\mathcal{C}_s^{gg,[1]}&=(\boldsymbol{T}^{a_3}\boldsymbol{T}^{a_4})_{i_1i_2}\ ,\\ \mathcal{C}_s^{gg,[2]}&=(\boldsymbol{T}^{a_4}\boldsymbol{T}^{a_3})_{i_1i_2}\ ,\\ \mathcal{C}_s^{gg,[3]}&=\boldsymbol{\delta}^{a_3a_4}\delta_{i_1i_2}\,,\end{aligned} \tag{4.27}$$

while for the processes with photons we have,

$$\begin{aligned}\mathcal{C}_s^{g\gamma,[1]} &= (\boldsymbol{T}^{a_3})_{i_1 i_2}\,,\\ \mathcal{C}_s^{\gamma\gamma,[1]} &= \delta_{i_1 i_2}\,,\end{aligned}\tag{4.28}$$

The u- and t-channel colour factors are obtained by crossing symmetry,

$$\mathcal{C}_u^{\mathcal{P},[i]} = \mathcal{C}_s^{\mathcal{P},[i]} \quad (1 \leftrightarrow 3)\,, \qquad \mathcal{C}_t^{\mathcal{P},[i]} = \mathcal{C}_s^{\mathcal{P},[i]} \quad (1 \leftrightarrow 4)\,. \tag{4.29}$$

There is considerable freedom in the definition of the helicity dependent spinor factors. For the s-channel, for process (4.15) we choose,

$$\begin{aligned}\mathcal{S}_{s+-+-}^{gg,[1]} &= -i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,,\\ \mathcal{S}_{s+-+-}^{gg,[2]} &= -i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}\,,\\ \mathcal{S}_{s+-+-}^{gg,[3]} = \mathcal{S}_{+-+-}^{g\gamma} = \mathcal{S}_{+-+-}^{\gamma\gamma} &= -i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle}\,,\end{aligned}\tag{4.30}$$

while for process (4.14) we choose,

$$\mathcal{S}_{s+-++}^{\mathcal{P},[i]} = i\frac{\langle 23\rangle[13]}{\langle 34\rangle^2}\,. \tag{4.31}$$

The factors for the other s-channel processes, (4.16), (4.17) are obtained by Bose symmetry (exchanging labels 3 and 4). Note that for the two-gluon process, this symmetry also exchanges the first two colour structures, [1] and [2]. The spinor prefactors for the u- and $t-$ channels are also obtained by crossing symmetry,

$$\mathcal{S}_{u\lambda_2\lambda_3\lambda_1\lambda_4}^{\mathcal{P},[i]} = \mathcal{S}_{s\lambda_2\lambda_1\lambda_3\lambda_4}^{\mathcal{P},[i]}\,(p_1 \leftrightarrow p_3)\,, \qquad \mathcal{S}_{t\lambda_2\lambda_4\lambda_3\lambda_1}^{\mathcal{P},[i]} = \mathcal{S}_{s\lambda_2\lambda_1\lambda_3\lambda_4}^{\mathcal{P},[i]}\,(p_1 \leftrightarrow p_4)\,. \tag{4.32}$$

4.2 Tree level helicity amplitudes

At tree-level, inserting the results for the tensor coefficients for the various processes given in section 3.1, the only non-vanishing helicity amplitudes in the s-channel are,

$$\begin{aligned}|\widehat{\mathcal{M}}_s^{gg,(0),[1]}\rangle_{+-+-} &= |\widehat{\mathcal{M}}_s^{gg,(0),[1]}\rangle_{+--+} = -2\,,\\ |\widehat{\mathcal{M}}_s^{gg,(0),[2]}\rangle_{+-+-} &= |\widehat{\mathcal{M}}_s^{gg,(0),[2]}\rangle_{+--+} = -2\,,\\ |\widehat{\mathcal{M}}_s^{g\gamma,(0)}\rangle_{+-+-} &= |\widehat{\mathcal{M}}^{g\gamma,(0)}\rangle_{+--+} = 2\,,\\ |\widehat{\mathcal{M}}_s^{\gamma\gamma,(0)}\rangle_{+-+-} &= |\widehat{\mathcal{M}}^{\gamma\gamma,(0)}\rangle_{+--+} = 2\,.\end{aligned}\tag{4.33}$$

The appropriate spinor factors are obtained from eq. (4.30) while the colour factors are given in eqs. (4.27) and (4.28). Tree helicity amplitudes for the other channels are obtained by the appropriate crossing symmetry.

4.3 One-loop helicity amplitudes

The one-loop helicity amplitudes are straightforwardly obtained by inserting the all-orders results for the tensor coefficients given in section 3.2 into eq. (4.6), expanding the one-loop bubble and box integrals around $\epsilon = 0$, and renormalising according to eq. (4.10). Explicit expansions in ϵ for the $d = 4 - 2\epsilon$ one-loop bubble and $d = 6 - 2\epsilon$ one-loop box graphs can be found, for example, in the appendix of ref. [1].

The finite remainders of the one-loop amplitudes defined through eq. (4.11) for the two-gluon process can be decomposed according to the number of colours and massless quark flavours,

$$|\widehat{\mathcal{M}}_c^{gg,(1),[i],\text{finite}}\rangle = N A_c^{gg,(1),[i]} + \frac{1}{N} B_c^{gg,(1),[i]} + N_F C_c^{gg,(1),[i]} - \beta_0 S \,|\widehat{\mathcal{M}}_c^{gg,(0),[i]}\rangle\,, \qquad (4.34)$$

for $i = 1, 2$ and

$$|\widehat{\mathcal{M}}_c^{gg,(1),[3],\text{finite}}\rangle = D_c^{gg,(1),[3]} + \frac{N_F}{N} E_c^{gg,(1),[3]} \qquad (4.35)$$

for $i = 3$ and where

$$S = \log\left(\frac{s}{\mu^2}\right) \qquad (4.36)$$

and β_0 is defined in eq. (2.6). For the other two processes we have,

$$|\widehat{\mathcal{M}}_c^{g\gamma,(1),[1],\text{finite}}\rangle = N A_c^{g\gamma,(1),[1]} + \frac{1}{N} B_c^{g\gamma,(1),[1]} + N_F C_c^{g\gamma,(1),[1]} - \frac{1}{2}\beta_0 S \,|\widehat{\mathcal{M}}_c^{g\gamma,(0),[1]}\rangle\,, \qquad (4.37)$$

$$|\widehat{\mathcal{M}}_c^{\gamma\gamma,(1),[1],\text{finite}}\rangle = N A_c^{\gamma\gamma,(1),[1]} + \frac{1}{N} B_c^{\gamma\gamma,(1),[1]} + N_F C_c^{\gamma\gamma,(1),[1]}\,. \qquad (4.38)$$

For clarity, the dependence on the helicities has been suppressed. Explicit expressions for $A, \ldots, E$ for $q\bar{q} \to gg$ in the physical region, $s = s_{12} > 0$, $t = s_{23} < 0$ and $u = s_{13} < 0$, are given in appendix B. Explicit results for $q\bar{q} \to gg$ scattering in the physical region, $s = s_{12} > 0$, $t = s_{23} < 0$ and $u = s_{13} < 0$ are given in appendix B. Formulae for the other channels and processes are available from the authors as FORM outputs. We have checked that the finite one-loop helicity amplitudes presented here are in agreement with the helicity amplitudes for $q\bar{q} \to gg$ (and crossed processes) given for the HV scheme in ref. [3].

4.4 Two-loop helicity amplitudes

The main results of this paper are the two-loop amplitudes for processes (4.14)–(4.25). As in the tree and one-loop cases, the helicity amplitudes can be directly extracted from the appropriate linear combination, eq. (4.6), of unrenormalised two-loop tensor coefficients obtained by direct evaluation of the projectors given in section 2.2 acting on two-loop graphs. Renormalisation is achieved via eq. (4.10) and the two-loop master integrals expanded around $\epsilon = 0$. The finite remainder of the two-loop amplitudes are defined through eq. (4.12) and, for each process can be decomposed according to the number of colours and massless quark flavours,

$$|\widehat{\mathcal{M}}_c^{gg,(2),[i],\text{finite}}\rangle = N^2 A_c^{gg,(2),[i]} + B_c^{gg,(2),[i]} + \frac{1}{N^2} C_c^{gg,(2),[i]} + N N_F D_c^{gg,(2),[i]} +$$

$$
+\frac{N_F}{N}E_c^{gg,(2),[i]} + N_F^2 F_c^{gg,(2),[i]} + \frac{N_F^2}{N^2}G_c^{gg,(2),[i]} -
$$
$$
-2\beta_0 S\,|\widehat{\mathcal{M}}_c^{gg,(1),[i],\mathrm{finite}}\rangle - (\beta_1 S + \beta_0^2 S^2)\,|\widehat{\mathcal{M}}_c^{gg,(0),[i]}\rangle, \tag{4.39}
$$

for $i = 1, 2$ and

$$
\begin{aligned}
|\widehat{\mathcal{M}}_c^{gg,(2),[3],\mathrm{finite}}\rangle &= N H_c^{gg,(2),[3]} + \frac{1}{N} I_c^{gg,(2),[3]} + N_F J_c^{gg,(2),[3]} + \\
&\quad + \frac{N_F}{N^2} K_c^{gg,(2),[3]} + \frac{N_F^2}{N} L_c^{gg,(2),[3]} + \\
&\quad -2\beta_0 S\,|\widehat{\mathcal{M}}_c^{gg,(1),[3],\mathrm{finite}}\rangle - (\beta_1 S + \beta_0^2 S^2)\,|\widehat{\mathcal{M}}_c^{gg,(0),[3]}\rangle\,.
\end{aligned} \tag{4.40}
$$

For the other processes we have,

$$
\begin{aligned}
|\widehat{\mathcal{M}}_c^{g\gamma,(2),[1],\mathrm{finite}}\rangle &= N^2 A_c^{g\gamma,(2),[1]} + B_c^{g\gamma,(2),[1]} + \frac{1}{N^2}\,C_c^{g\gamma,(2),[1]} + N N_F D_c^{g\gamma,(2),[1]} + \\
&\quad + \frac{N_F}{N}\,E_c^{g\gamma,(2),[1]} + N_F^2\,F_c^{g\gamma,(2),[1]} + \left(\sum_q Q_q\right)\left(\frac{4}{N} - N\right)\,G_c^{g\gamma,(2),[1]} - \\
&\quad - \frac{3}{2}\beta_0 S\,|\widehat{\mathcal{M}}_c^{g\gamma,(1),[1],\mathrm{finite}}\rangle - \left(\frac{1}{2}\beta_1 S + \frac{3}{8}\beta_0^2 S^2\right)|\widehat{\mathcal{M}}_c^{g\gamma,(0),[1]}\rangle\,,
\end{aligned} \tag{4.41}
$$

$$
\begin{aligned}
|\widehat{\mathcal{M}}_c^{\gamma\gamma,(2),[1],\mathrm{finite}}\rangle &= N^2 A_c^{\gamma\gamma,(2),[1]} + B_c^{\gamma\gamma,(2),[1]} + \frac{1}{N^2}\,C_c^{\gamma\gamma,(2),[1]} + N N_F D_c^{\gamma\gamma,(2),[1]} + \\
&\quad + \frac{N_F}{N}\,E_c^{\gamma\gamma,(2),[1]} + \left(\sum_q Q_q^2\right) C_F\,F_c^{\gamma\gamma,(2),[1]} - \beta_0 S\,|\widehat{\mathcal{M}}_c^{\gamma\gamma,(1),[1],\mathrm{finite}}\rangle\,,
\end{aligned} \tag{4.42}
$$

where Q_q is the electric charge of a massless quark of flavour q.

As in the previous section, the dependence on the helicities has been suppressed. Explicit expressions for $A, \ldots, L$ for $q\bar{q} \to gg$ process in the physical region, $s = s_{12} > 0$, $t = s_{23} < 0$ and $u = s_{13} < 0$, are given in appendix A. Formulae for the other channels and processes are available from the authors as FORM outputs. We have checked that the finite two-loop helicity amplitudes presented here are in agreement with the helicity amplitudes for $q\bar{q} \to gg$ (and crossed processes) given for the HV scheme in ref. [3].

5. Comparison with previous results

In recovering the square, or interference, of amplitudes in a particular channel c, the contraction of the n-loop colour vector $|\mathcal{M}^{\mathcal{P},(n)}\rangle$ with a conjugate m-loop colour vector $\langle\mathcal{M}^{\mathcal{P},(m)}|$ obeys the rule

$$
\langle\mathcal{M}^{\mathcal{P},(m)}|\mathcal{M}^{\mathcal{P},(n)}\rangle = \sum_{\text{helicities}}\ \sum_{\text{colours}} |\mathcal{M}_c^{\mathcal{P},(n)}\rangle^*_{\lambda_1\lambda_2\lambda_3\lambda_4}\,|\mathcal{M}_c^{\mathcal{P},(n)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4} \tag{5.1}
$$

where $|\mathcal{M}_c^{\mathcal{P},(n)}\rangle_{\lambda_1\lambda_2\lambda_3\lambda_4}$ contains all of the spinor and colour information for a particular helicity. Expanding the colour and spinor factors $\mathcal{C}$ and $\mathcal{S}$, and dropping the explicit dependence on the helicities, we see that,

$$
\langle\mathcal{M}^{\mathcal{P},(m)}|\mathcal{M}^{\mathcal{P},(n)}\rangle = \sum_{\text{helicities}}\ \sum_{\text{colours}}\ \sum_{i,j} \mathcal{C}_c^{\mathcal{P},[i]*}\,\mathcal{C}_c^{\mathcal{P},[j]}\,\mathcal{S}_c^{\mathcal{P},[i]*}\,\mathcal{S}_c^{\mathcal{P},[j]}\,|\widehat{\mathcal{M}}_c^{\mathcal{P},(n),[i]}\rangle^*\,|\widehat{\mathcal{M}}_c^{\mathcal{P},(n),[j]}\rangle
$$

$$= \sum_{\text{helicities}} \sum_{i,j} \mathcal{CC}^{\mathcal{P}}_{ij}\, \mathcal{SS}^{\mathcal{P}}_{ij}\, |\widehat{\mathcal{M}}^{\mathcal{P},(n),[i]}_c\rangle^* \, |\widehat{\mathcal{M}}^{\mathcal{P},(n),[j]}_c\rangle \,. \tag{5.2}$$

The symmetric colour sum matrix $\mathcal{CC}^{\mathcal{P}} = \sum_{\text{colours}} \mathcal{C}^{\mathcal{P},[i]*}_c \, \mathcal{C}^{\mathcal{P},[j]}_c$ is given by

$$\mathcal{CC}^{gg} = \frac{N^2-1}{2} \begin{pmatrix} \frac{N^2-1}{2N} & -\frac{1}{2N} & 1 \\ -\frac{1}{2N} & \frac{N^2-1}{2N} & 1 \\ 1 & 1 & 2N \end{pmatrix}, \tag{5.3}$$

and,

$$\mathcal{CC}^{g\gamma} = \frac{N^2-1}{2}, \qquad \mathcal{CC}^{\gamma\gamma} = N \,. \tag{5.4}$$

In the s-channel, the process and helicity dependent spinor matrices $\mathcal{SS}^{\mathcal{P}}_c = \mathcal{S}^{\mathcal{P},[i]*}_c \, \mathcal{S}^{\mathcal{P},[j]}_c$ are given by,

$$\mathcal{SS}^{gg}_{s+-+-} = \frac{u}{s^2 t} \begin{pmatrix} u^2 & ut & us \\ tu & t^2 & ts \\ su & ts & s^2 \end{pmatrix}, \qquad \mathcal{SS}^{g\gamma}_{s+-+-} = \mathcal{SS}^{\gamma\gamma}_{s+-+-} = \frac{u}{t}\,, \tag{5.5}$$

and

$$\mathcal{SS}^{gg}_{s+-++} = \frac{ut}{s^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathcal{SS}^{g\gamma}_{s+-++} = \mathcal{SS}^{\gamma\gamma}_{s+-++} = \frac{ut}{s^2}\,. \tag{5.6}$$

Similar matrices for the other helicities are obtained by Bose symmetry, while corresponding matrices for the u and t-channels are obtained by crossing symmetry,

$$\mathcal{SS}^{\mathcal{P}}_u = -\mathcal{SS}^{\mathcal{P}}_s(s \leftrightarrow t)\,, \qquad \mathcal{SS}^{\mathcal{P}}_t = -\mathcal{SS}^{\mathcal{P}}_s(s \leftrightarrow u)\,. \tag{5.7}$$

The explicit minus sign accounts for the exchange of a single fermion between the initial and final states.

Eqs. (5.1) and (5.2) can be used to recreate the full interference of tree and two-loop amplitudes. However, we notice that we can immediately separate the singularities from the finite parts using eq. (4.12). The finite remainder, denoted by $\mathcal{F}^{2\times 0}_{inite}(s,t,u)$ in refs. [1, 2] is thus given by,

$$2\,\mathrm{Re} \sum_{\text{helicities}} \sum_{i,j} \mathcal{CC}^{\mathcal{P}}_{ij}\, \mathcal{SS}^{\mathcal{P}}_{ij}\, |\widehat{\mathcal{M}}^{\mathcal{P},(0),[i]}_c\rangle^* \, |\widehat{\mathcal{M}}^{\mathcal{P},(2),[j],\text{finite}}_c\rangle \,. \tag{5.8}$$

Note that by using helicities, we are implicitly treating the external particle states in 4-dimensions - the HV scheme. However, the singular contributions to the renormalised amplitudes defined in eqs. (4.11), (4.12) are given relative to the tree amplitude and therefore any additional terms that would be produced in CDR are automatically removed. Treating the external states differently for both the n-loop and tree amplitudes does not alter the finite contribution. Direct application of eq. (5.1) for the real part of the tree and two-loop interference ($n = 0$ and $m = 2$) in the HV scheme therefore generates the same finite parts given in section 3.2 of ref. [1] for quark-gluon scattering and in section 4 and appendix A of ref. [2] obtained using CDR.

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Similarly, the square of one-loop graphs is obtained using $n = 1$ and $m = 1$ in eq. (5.1). In principle, the one-loop amplitude should be expanded through to $\mathcal{O}(\epsilon^2)$. However, as noted in [1, 30, 64], the finite self interference only contains logarithms (up to the fourth power), but does not contain triple and quartic polylogs. These terms naturally arise when the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms in the expansion of the one-loop box are multiplied by singularities from the conjugate amplitude. Such terms appear in the Catani pole structure, $\langle\mathcal{M}^{(1)}|\boldsymbol{I}^{(1)}|\mathcal{M}^{(0)}\rangle$ where $|\mathcal{M}^{(1)}\rangle$ does need to be expanded through to $\mathcal{O}(\epsilon^2)$. The finite parts of the renormalised one-loop self interference remaining after Catani's prediction for the singularities has been removed should therefore be identical to those obtained by summing the finite parts using eq. (5.1). In this instance, the finite remainder denoted by $\mathcal{F}^{1\times 1}_{inite}(s,t,u)$ in [2] is obtained from

$$\sum_{\text{helicities}} \sum_{i,j} \mathcal{CC}^{\mathcal{P}}_{ij}\, \mathcal{SS}^{\mathcal{P}}_{ij}\, |\widehat{\mathcal{M}}^{\mathcal{P},(1),[i],\text{finite}}_c\rangle^* \, |\widehat{\mathcal{M}}^{\mathcal{P},(1),[j],\text{finite}}_c\rangle. \tag{5.9}$$

We have checked that this procedure correctly reproduces the coefficients given in section 5 and appendix B of ref. [2] for the processes involving one or more photons. Unfortunately, the results of ref. [1] are not presented in quite the same way and a direct comparison is more difficult.[2] However, we have checked the finite remainders after the Catani poles have been subtracted, both directly from the square of the one-loop amplitude (the CDR scheme) and the one-loop square reconstructed from the helicity amplitudes (the HV scheme) with the HV results of ref. [3] and again find agreement.

6. Conclusions

In this paper we have presented the analytic expressions for the one- and two-loop virtual helicity amplitudes for quark-gauge boson scattering, $q\bar{q} \to gg$, $q\bar{q} \to g\gamma$ and $q\bar{q} \to \gamma\gamma$ and those processes related by crossing symmetry. These amplitudes were obtained by the construction of d-dimensional projection operators that extract the coefficients of the most general tensor representation for the amplitude, order by order in perturbation theory. Once the renormalised tensor coefficients are known, the renormalised helicity amplitudes can be constructed straightforwardly as a linear combination of the coefficients.

At the level of the diagrammatic calculation, we applied the projection operators directly to the Feynman diagrams that contribute at each order in α_s. The projectors saturate the tensor structure of the Feynman diagram and yield a set of scalar integrals that can be related to a basis set of master integrals with the application of widely used reduction algorithms. In fact, we are using exactly the same tools as we did in [1, 2], except in that case, the projector was merely the conjugate tree amplitude. Because the projectors exist in d-dimensions, conventional dimensional regularisation is preserved and there is no ambiguity in dealing with γ_5. Once the general tensor coefficients are determined, the external particles can be treated in 4-dimensions (the 't Hooft-Veltman scheme) and standard helicity methods used to construct the helicity amplitudes.

[2] In fact there is a misprint in the renormalisation term, $\mathcal{R}_{\mathcal{A}}$, of eq. 4.6 of ref. [1].

By summing over helicities and colours, the full interference of tree and two-loop graphs can be reconstructed. In previous work [1, 2], the finite part has been separated from infrared singular parts using the Catani formalism [49]. In this procedure, the singular operators multiply tree and one-loop amplitudes. Changing the scheme for the external particles therefore changes the overall pole contribution. However, the finite remainder left after subtracting the poles is invariant under changing from the conventional dimensional regularisation scheme to the 't Hooft-Veltman scheme. Therefore, the finite parts of the helicity amplitudes presented here precisely reproduce the finite parts of the interference of tree and two-loop graphs given in [1, 2]. Similarly, the finite parts of the self interference of one-loop amplitudes can be reconstructed purely from the finite parts of the one-loop helicity amplitudes. In addition, the finite remainders of the two-loop amplitudes for quark-gluon scattering in the HV scheme agree with the independent calculation of ref. [3].

For the case of quark-gluon scattering, the presence of four coloured external particles gives rise to additional colour correlations proportional to $1/\epsilon$. While the existence of such additional non-trivial colour correlations was pointed out in ref. [49] and are expected on general grounds [57], the precise form of the colour structure of $\boldsymbol{H}^{\mathcal{P},(2)}$ was not predicted. So far, the results for gluon-gluon scattering [30] as well as those for quark-gluon scattering presented in this paper and in ref. [3], support the form given in eqs. (3.38) and (3.39). These colour correlations vanish when the interference of tree and two-loop amplitudes is constructed.

The more general goal is to evaluate the NNLO QCD corrections to the associated scattering processes which should yield a significant improvement in theoretical stability, see for example [4]. However, the reality is that the phenomenology is still some way off and there is still a significant amount of work remaining to be done. The two-loop corrections to the $2 \to 2$ partonic process presented here, need to be combined with infrared singular contributions from both the one-loop $2 \to 3$ partonic process, when one of the partons becomes unresolved [65, 66, 63] and the tree-level $2 \to 4$ partonic process when two partons become unresolved [67]–[71]. These different contributions need to be put together in such a way to make the explicit singularities analytically cancel, so that the four-dimensional limit can be safely taken, and then by encoding the finite remainders in a stable numerical implementation. There has been recent progress in identifying appropriate counterterms [72, 73] and we are optimistic that the technical problems will soon be overcome enabling predictions of both dijet, jet and photon and diphoton production at the Tevatron and the LHC at an accuracy of $\mathcal{O}(5\ )$ which is expected to be competitive with the experimental measurements.

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A. Finite two-loop contributions

In this appendix we give explicit formulae for the coefficients $A, \ldots, L$ for the $q\bar{q} \to gg$ process. Expressions for quark-gluon scattering as well as the processes involving photons are available from the authors as a FORM output file.

As usual, the polylogarithms $\mathrm{Li}_n(w)$ are defined by

$$\begin{aligned}
\mathrm{Li}_n(w) &= \int_0^w \frac{dt}{t}\mathrm{Li}_{n-1}(t) \qquad \text{for } n = 2,3,4 \\
\mathrm{Li}_2(w) &= -\int_0^w \frac{dt}{t}\log(1-t)\,.
\end{aligned} \tag{A.1}$$

Using the standard polylogarithm identities [74], we retain the polylogarithms with arguments x, $1-x$ and $(x-1)/x$, where

$$x = -\frac{t}{s}\,, \qquad y = -\frac{u}{s} = 1 - x\,, \qquad z = -\frac{u}{t} = \frac{x-1}{x}\,. \tag{A.2}$$

For convenience, we also introduce the following logarithms

$$X = \log\left(\frac{-t}{s}\right), \qquad Y = \log\left(\frac{-u}{s}\right), \qquad S = \log\left(\frac{s}{\mu^2}\right), \tag{A.3}$$

where μ is the renormalisation scale.

A.1 $q(p_2,+) + \bar{q}(p_1,-) \to g(p_3,+) + g(p_4,+)$

$$\begin{aligned}
A^{gg,(2),[1]}_{s+-++} &= \left(-\frac{25}{24}X^2 - \frac{35}{18}X - \frac{175}{54}\right) + \left(-\frac{17}{8}X^2 - \frac{5}{2}X\right)\frac{t}{u} - \frac{5}{4}X^2\frac{t^2}{u^2} + \\
&\quad + \left(-\frac{1}{8}X^2 + \frac{11}{8}X - \frac{475}{144}\right)\frac{s}{t} + \\
&\quad + \left(\left(-\frac{25}{12}X - \frac{19}{6}\right) + \left(-\frac{17}{4}X - \frac{5}{2}\right)\frac{t}{u} - \frac{5}{2}X\frac{t^2}{u^2} + \left(-\frac{1}{4}X - \frac{11}{24}\right)\frac{s}{t}\right)i\pi
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
B^{gg,(2),[1]}_{s+-++} &= \left(\frac{1}{4}X^2 - \frac{7}{4}XY + \frac{1}{2}X + \frac{7}{8}Y^2 - \frac{3}{2}Y + \frac{7}{8}\pi^2 - \frac{2}{3}\right) + \left(-\frac{9}{8}X^2 - X\right)\frac{t}{u} - \\
&\quad - \frac{1}{2}X^2\frac{t^2}{u^2} + \left(\frac{3}{4}X^2 - \frac{3}{2}XY + \frac{3}{4}Y^2 + \frac{3}{4}\pi^2\right)\frac{t}{s} + \\
&\quad + \left(\frac{7}{4}X + \frac{1}{4}Y^2 - \frac{3}{4}Y - \frac{139}{36}\right)\frac{s}{t} + \frac{1}{8}Y^2\frac{s^2}{t^2} + \\
&\quad + \left(\left(-\frac{5}{4}X - 1\right) + \left(-\frac{9}{4}X - 1\right)\frac{t}{u} - X\frac{t^2}{u^2} + \left(\frac{1}{2}Y - \frac{5}{6}\right)\frac{s}{t} + \frac{1}{4}Y\frac{s^2}{t^2}\right)i\pi
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
C^{gg,(2),[1]}_{s+-++} &= \left(\frac{7}{8}X^2 - \frac{7}{4}XY + 2X + \frac{7}{8}Y^2 - \frac{3}{2}Y + \frac{7}{8}\pi^2\right) + \left(\frac{1}{4}X^2 + \frac{1}{2}X\right)\frac{t}{u} + \\
&\quad + \frac{1}{4}X^2\frac{t^2}{u^2} + \left(\frac{3}{4}X^2 - \frac{3}{2}XY + \frac{3}{4}Y^2 + \frac{3}{4}\pi^2\right)\frac{t}{s} + \\
&\quad + \left(\frac{1}{8}X^2 + \frac{3}{8}X + \frac{1}{4}Y^2 - \frac{3}{4}Y - \frac{17}{16}\right)\frac{s}{t} + \frac{1}{8}Y^2\frac{s^2}{t^2} +
\end{aligned}$$

$$
+\left(\frac{1}{2}+\left(\frac{1}{2}X+\frac{1}{2}\right)\frac{t}{u}+\frac{1}{2}X\frac{t^2}{u^2}+\left(\frac{1}{4}X+\frac{1}{2}Y-\frac{3}{8}\right)\frac{s}{t}+\frac{1}{4}Y\frac{s^2}{t^2}\right)i\pi \tag{A.6}
$$

$$
\begin{aligned}
D^{gg,(2),[1]}_{s+-++} = & \left(\frac{5}{12}X^2+\frac{7}{18}X+\frac{397}{108}\right)+\left(\frac{3}{4}X^2+X\right)\frac{t}{u}+\frac{1}{2}X^2\frac{t^2}{u^2}+\left(-\frac{1}{4}X+\frac{37}{36}\right)\frac{s}{t}+ \\
& +\left(\left(\frac{5}{6}X+\frac{11}{6}\right)+\left(\frac{3}{2}X+1\right)\frac{t}{u}+X\frac{t^2}{u^2}+\frac{1}{12}\frac{s}{t}\right)i\pi
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
E^{gg,(2),[1]}_{s+-++} = & \left(-\frac{1}{2}XY+\frac{3}{2}X+\frac{1}{4}Y^2-3Y+\frac{1}{4}\pi^2+\frac{11}{12}\right)+\left(-X^2-\frac{3}{2}X\right)\frac{t}{u}- \\
& -\frac{3}{4}X^2\frac{t^2}{u^2}+\left(\frac{1}{2}X^2-XY+\frac{1}{2}Y^2+\frac{1}{2}\pi^2\right)\frac{t}{s}+ \\
& +\left(-\frac{1}{4}X+\frac{1}{2}Y^2-\frac{3}{2}Y+\frac{19}{36}\right)\frac{s}{t}+\frac{3}{4}Y^2\frac{s^2}{t^2}+ \\
& +\left(\left(-\frac{1}{2}X-\frac{3}{2}\right)+\left(-2X-\frac{3}{2}\right)\frac{t}{u}-\frac{3}{2}X\frac{t^2}{u^2}+\left(Y-\frac{17}{12}\right)\frac{s}{t}+\frac{3}{2}Y\frac{s^2}{t^2}\right)i\pi
\end{aligned} \tag{A.8}
$$

$$
F^{gg,(2),[1]}_{s+-++} = -\frac{1}{6}i\pi+\left(\frac{1}{18}X-\frac{5}{27}\right) \tag{A.9}
$$

$$
G^{gg,(2),[1]}_{s+-++} = 0 \tag{A.10}
$$

$$
\begin{aligned}
A^{gg,(2),[2]}_{s+-++} = & \left(\frac{1}{24}Y^2+\frac{59}{72}Y-\frac{25}{432}\right)+\left(-\frac{1}{8}Y^2+\frac{11}{8}Y-\frac{475}{144}\right)\frac{t}{u}+\left(\frac{3}{8}Y^2-\frac{5}{2}Y\right)\frac{s}{t}+ \\
& +\frac{5}{4}Y^2\frac{s^2}{t^2}+\left(\left(\frac{1}{12}Y+\frac{5}{24}\right)+\left(-\frac{1}{4}Y-\frac{11}{24}\right)\frac{t}{u}+\left(\frac{3}{4}Y-\frac{5}{2}\right)\frac{s}{t}+\frac{5}{2}Y\frac{s^2}{t^2}\right)i\pi
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
B^{gg,(2),[2]}_{s+-++} = & \left(\frac{1}{4}XY+\frac{3}{4}X-\frac{1}{8}Y^2+\frac{1}{4}Y-\frac{1}{8}\pi^2-\frac{115}{36}\right)+\left(-\frac{3}{4}X+\frac{7}{4}Y-\frac{139}{36}\right)\frac{t}{u}- \\
& -\frac{1}{8}X^2\frac{t^2}{u^2}+\left(\frac{3}{4}X^2-\frac{3}{2}XY+\frac{3}{4}Y^2+\frac{3}{4}\pi^2\right)\frac{t}{s}+\left(-\frac{1}{8}Y^2-Y\right)\frac{s}{t}+\frac{1}{2}Y^2\frac{s^2}{t^2}+ \\
& +\left(\left(\frac{1}{4}X-\frac{5}{6}\right)-\frac{5}{6}\frac{t}{u}-\frac{1}{4}X\frac{t^2}{u^2}+\left(-\frac{1}{4}Y-1\right)\frac{s}{t}+Y\frac{s^2}{t^2}\right)i\pi
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
C^{gg,(2),[2]}_{s+-++} = & \left(\frac{1}{4}XY+\frac{3}{4}X-\frac{9}{8}Y-\frac{1}{8}\pi^2-\frac{17}{16}\right)+\left(-\frac{3}{4}X+\frac{1}{8}Y^2+\frac{3}{8}Y-\frac{17}{16}\right)\frac{t}{u}- \\
& -\frac{1}{8}X^2\frac{t^2}{u^2}+\left(\frac{3}{4}X^2-\frac{3}{2}XY+\frac{3}{4}Y^2+\frac{3}{4}\pi^2\right)\frac{t}{s}+ \\
& +\left(-\frac{1}{4}Y^2+\frac{1}{2}Y\right)\frac{s}{t}-\frac{1}{4}Y^2\frac{s^2}{t^2}+ \\
& +\left(\left(\frac{1}{4}X+\frac{1}{4}Y-\frac{3}{8}\right)+\left(\frac{1}{4}Y-\frac{3}{8}\right)\frac{t}{u}-\frac{1}{4}X\frac{t^2}{u^2}+\left(-\frac{1}{2}Y+\frac{1}{2}\right)\frac{s}{t}-\frac{1}{2}Y\frac{s^2}{t^2}\right)i\pi
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
D^{gg,(2),[2]}_{s+-++} = & \left(-\frac{1}{6}Y^2+\frac{13}{36}Y-\frac{143}{54}\right)+\left(-\frac{1}{4}Y+\frac{37}{36}\right)\frac{t}{u}+\left(-\frac{1}{4}Y^2+Y\right)\frac{s}{t}-\frac{1}{2}Y^2\frac{s^2}{t^2}+ \\
& +\left(\left(-\frac{1}{3}Y-\frac{3}{4}\right)+\frac{1}{12}\frac{t}{u}+\left(-\frac{1}{2}Y+1\right)\frac{s}{t}-Y\frac{s^2}{t^2}\right)i\pi
\end{aligned} \tag{A.14}
$$

$$
E^{gg,(2),[2]}_{s+-++} = \left(-\frac{1}{2}XY+\frac{3}{2}X+\frac{1}{4}Y^2-\frac{13}{4}Y+\frac{1}{4}\pi^2-\frac{7}{18}\right)+
$$

$$\begin{aligned}
&+\left(-X^2 - \frac{3}{2}X - \frac{1}{4}Y + \frac{19}{36}\right)\frac{t}{u} - \frac{3}{4}X^2\frac{t^2}{u^2} + \left(\frac{1}{2}X^2 - XY + \frac{1}{2}Y^2 + \frac{1}{2}\pi^2\right)\frac{t}{s} + \\
&+\left(\frac{1}{2}Y^2 - \frac{3}{2}Y\right)\frac{s}{t} + \frac{3}{4}Y^2\frac{s^2}{t^2} + \\
&+\left(\left(-\frac{1}{2}X - \frac{17}{12}\right) + \left(-2X - \frac{17}{12}\right)\frac{t}{u} - \frac{3}{2}X\frac{t^2}{u^2} + \left(Y - \frac{3}{2}\right)\frac{s}{t} + \frac{3}{2}Y\frac{s^2}{t^2}\right)i\pi
\end{aligned}\tag{A.15}$$

$$F^{gg,(2),[2]}_{s+-++} = \frac{1}{6}i\pi + \left(-\frac{1}{18}Y + \frac{5}{27}\right)\tag{A.16}$$

$$G^{gg,(2),[2]}_{s+-++} = 0\tag{A.17}$$

$$\begin{aligned}
H^{gg,(2),[3]}_{s+-++} = &\left(-\frac{1}{48}X^2 - \frac{1}{2}XY + \frac{11}{18}X + \frac{5}{24}Y^2 - \frac{16}{9}Y + \frac{1}{4}\pi^2\right) + \\
&+\left(-\frac{11}{16}X^2 - \frac{1}{4}XY - \frac{7}{6}X + \frac{1}{8}\pi^2\right)\frac{t}{u} - \frac{5}{8}X^2\frac{t^2}{u^2} + \frac{5}{8}Y^2\frac{s^2}{t^2} + \\
&+\left(\frac{1}{4}X^2 - \frac{1}{2}XY + \frac{1}{4}Y^2 + \frac{1}{4}\pi^2\right)\frac{t}{s} + \left(-\frac{1}{4}XY + \frac{9}{16}Y^2 - \frac{7}{6}Y + \frac{1}{8}\pi^2\right)\frac{s}{t} + \\
&+\left(\left(-\frac{13}{24}X - \frac{1}{12}Y - \frac{7}{6}\right) + \left(-\frac{13}{8}X - \frac{1}{4}Y - \frac{7}{6}\right)\frac{t}{u} - \frac{5}{4}X\frac{t^2}{u^2} + \right. \\
&\left. + \left(-\frac{1}{4}X + \frac{7}{8}Y - \frac{7}{6}\right)\frac{s}{t} + \frac{5}{4}Y\frac{s^2}{t^2}\right)i\pi
\end{aligned}\tag{A.18}$$

$$\begin{aligned}
I^{gg,(2),[3]}_{s+-++} = &\left(-\frac{1}{16}X^2 + \frac{1}{4}XY - \frac{11}{12}X - \frac{1}{4}Y^2 + Y - \frac{1}{8}\pi^2\right) + \\
&+\left(\frac{3}{16}X^2 - \frac{1}{4}XY + \frac{1}{12}X + \frac{1}{8}\pi^2\right)\frac{t}{u} + \\
&+\left(-\frac{1}{2}X^2 + XY - \frac{1}{2}Y^2 - \frac{1}{2}\pi^2\right)\frac{t}{s} + \\
&+\left(-\frac{1}{4}XY + \frac{3}{16}Y^2 + \frac{1}{12}Y + \frac{1}{8}\pi^2\right)\frac{s}{t} + \\
&+\left(\left(\frac{1}{8}X - \frac{1}{4}Y + \frac{1}{12}\right) + \left(\frac{1}{8}X - \frac{1}{4}Y + \frac{1}{12}\right)\frac{t}{u} + \left(-\frac{1}{4}X + \frac{1}{8}Y + \frac{1}{12}\right)\frac{s}{t}\right)i\pi
\end{aligned}\tag{A.19}$$

$$\begin{aligned}
J^{gg,(2),[3]}_{s+-++} = &\left(-\frac{1}{24}X^2 + \frac{1}{2}XY - \frac{55}{72}X - \frac{1}{3}Y^2 + \frac{47}{36}Y - \frac{1}{4}\pi^2\right) + \left(\frac{3}{8}X^2 + \frac{13}{24}X\right)\frac{t}{u} + \\
&+\frac{1}{4}X^2\frac{t^2}{u^2} + \left(-\frac{1}{2}X^2 + XY - \frac{1}{2}Y^2 - \frac{1}{2}\pi^2\right)\frac{t}{s} + \left(-\frac{1}{8}Y^2 + \frac{13}{24}Y\right)\frac{s}{t} - \frac{1}{4}Y^2\frac{s^2}{t^2} \\
&+\left(\left(\frac{5}{12}X - \frac{1}{6}Y + \frac{13}{24}\right) + \left(\frac{3}{4}X + \frac{13}{24}\right)\frac{t}{u} + \frac{1}{2}X\frac{t^2}{u^2} + \right. \\
&\quad \left. + \left(-\frac{1}{4}Y + \frac{13}{24}\right)\frac{s}{t} - \frac{1}{2}Y\frac{s^2}{t^2}\right)i\pi
\end{aligned}\tag{A.20}$$

$$\begin{aligned}
K^{gg,(2),[3]}_{s+-++} = &\left(\frac{1}{4}XY - \frac{17}{24}X - \frac{1}{8}Y^2 + \frac{3}{2}Y - \frac{1}{8}\pi^2\right) + \left(\frac{1}{2}X^2 + \frac{19}{24}X\right)\frac{t}{u} \\
&+\frac{3}{8}X^2\frac{t^2}{u^2} + \left(-\frac{1}{4}X^2 + \frac{1}{2}XY - \frac{1}{4}Y^2 - \frac{1}{4}\pi^2\right)\frac{t}{s} +
\end{aligned}$$

$$+\left(-\frac{1}{4}Y^2+\frac{19}{24}Y\right)\frac{s}{t}-\frac{3}{8}Y^2\frac{s^2}{t^2}+ \tag{A.21}$$
$$+\left(\left(\frac{1}{4}X+\frac{19}{24}\right)+\left(X+\frac{19}{24}\right)\frac{t}{u}+\frac{3}{4}X\frac{t^2}{u^2}+\left(-\frac{1}{2}Y+\frac{19}{24}\right)\frac{s}{t}-\frac{3}{4}Y\frac{s^2}{t^2}\right)i\pi$$

$$L^{gg,(2),[3]}_{s+-++}=\left(\frac{1}{36}X-\frac{1}{36}Y\right). \tag{A.22}$$

A.2 $q(p_2,+)+\bar{q}(p_1,-)\to g(p_3,+)+g(p_4,-)$

$$\begin{aligned}
A^{gg,(2),[1]}_{s+-+-} = {}&\Bigg(\frac{1}{2}\mathrm{Li}_4(x)+\frac{35}{12}\mathrm{Li}_3(x)-\frac{1}{4}\mathrm{Li}_2(x)X^2-\frac{35}{12}\mathrm{Li}_2(x)X+\frac{1}{3}\mathrm{Li}_2(x)\pi^2-\\
&-\frac{1}{16}X^4-\frac{1}{6}X^3Y+\frac{77}{72}X^3-\frac{35}{24}X^2Y+\frac{1}{3}X^2\pi^2-\frac{505}{144}X^2+\\
&+\frac{1}{3}XY\pi^2+2X\zeta_3+\frac{143}{72}X\pi^2-\frac{2273}{432}X-\frac{241}{24}\zeta_3-\frac{19}{288}\pi^4+\frac{5}{4}\pi^2+\frac{36077}{1728}\Bigg)+\\
&+\Bigg(-\frac{1}{4}\mathrm{Li}_3(x)+\frac{1}{4}\mathrm{Li}_2(x)X-\frac{7}{24}X^3+\frac{1}{8}X^2Y+\frac{31}{16}X^2+\frac{2}{3}X\pi^2+\\
&\quad+\frac{203}{144}X-\frac{1}{2}\zeta_3+\frac{17}{36}\pi^2+\frac{475}{144}\Bigg)\frac{t}{u}+\\
&+\Bigg(\frac{1}{2}\mathrm{Li}_3(x)-\frac{1}{2}\mathrm{Li}_2(x)X+\frac{1}{3}X^3-\frac{1}{4}X^2Y+\frac{11}{16}X^2-\frac{5}{6}X\pi^2+\frac{475}{72}X+\\
&\quad+\zeta_3-\frac{23}{18}\pi^2\Bigg)\frac{t^2}{u^2}+\\
&+\Bigg(\frac{3}{2}\mathrm{Li}_4(x)-\mathrm{Li}_3(x)X+\frac{1}{4}\mathrm{Li}_2(x)X^2+\frac{7}{48}X^4-\frac{43}{36}X^3-\frac{2}{3}X^2\pi^2+\\
&\quad+\frac{475}{144}X^2+X\zeta_3-\frac{23}{18}X\pi^2-\frac{1}{60}\pi^4\Bigg)\frac{t^3}{u^3}+\\
&+\Bigg(\Bigg(-\frac{1}{2}\mathrm{Li}_2(x)X-\frac{35}{12}\mathrm{Li}_2(x)-\frac{1}{4}X^3-\frac{1}{2}X^2Y+\frac{11}{8}X^2-\frac{35}{12}XY+\\
&\quad+\frac{1}{6}X\pi^2-\frac{7}{24}X+2\zeta_3+\frac{11}{24}\pi^2+\frac{823}{432}\Bigg)+\\
&\quad+\left(\frac{1}{4}\mathrm{Li}_2(x)-\frac{7}{8}X^2+\frac{1}{4}XY+\frac{49}{24}X+\frac{1}{12}\pi^2+\frac{467}{144}\right)\frac{t}{u}+\\
&\quad+\left(-\frac{1}{2}\mathrm{Li}_2(x)+X^2-\frac{1}{2}XY+\frac{121}{24}X-\frac{1}{6}\pi^2+\frac{475}{72}\right)\frac{t^2}{u^2}+\\
&\quad+\left(-\mathrm{Li}_3(x)+\frac{1}{2}\mathrm{Li}_2(x)X+\frac{7}{12}X^3-\frac{7}{4}X^2-\frac{1}{6}X\pi^2+\frac{475}{72}X+\zeta_3\right)\frac{t^3}{u^3}\Bigg)i\pi
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
B^{gg,(2),[1]}_{s+-+-} = {}&\Bigg(2\mathrm{Li}_4(x)-7\mathrm{Li}_4(y)+\mathrm{Li}_4(z)+\mathrm{Li}_3(x)X-\mathrm{Li}_3(x)Y+\frac{9}{2}\mathrm{Li}_3(x)+2\mathrm{Li}_3(y)X+\\
&+4\mathrm{Li}_3(y)Y+6\mathrm{Li}_3(y)-\frac{5}{2}\mathrm{Li}_2(x)X^2-\frac{9}{2}\mathrm{Li}_2(x)X+\frac{1}{6}\mathrm{Li}_2(x)\pi^2-4\mathrm{Li}_2(y)XY-\\
&-6\mathrm{Li}_2(y)Y+\frac{1}{6}X^4-\frac{11}{6}X^3Y-\frac{25}{36}X^3-\frac{3}{2}X^2Y^2+\frac{13}{12}X^2Y+\frac{3}{4}X^2\pi^2-\\
&-\frac{95}{36}X^2+\frac{5}{6}XY^3-\frac{13}{2}XY^2-\frac{1}{2}XY\pi^2+\frac{25}{9}XY-2X\zeta_3-\frac{40}{9}X\pi^2+\frac{97}{12}X-
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{12}Y^4+\frac{5}{3}Y^3-\frac{1}{4}Y^2\pi^2-\frac{191}{72}Y^2+Y\zeta_3+\frac{7}{6}Y\pi^2+\frac{3}{2}Y+\frac{11}{72}\zeta_3+\\
&+\frac{73}{1440}\pi^4-\frac{329}{144}\pi^2-\frac{14135}{1296}\Bigg)+\\
&+\Bigg(\frac{9}{2}\mathrm{Li}_4(x)-15\mathrm{Li}_4(y)-11\mathrm{Li}_4(z)+7\mathrm{Li}_3(x)X-7\mathrm{Li}_3(x)Y+\frac{13}{2}\mathrm{Li}_3(x)+\\
&\quad+14\mathrm{Li}_3(y)X+6\mathrm{Li}_3(y)-\frac{15}{4}\mathrm{Li}_2(x)X^2-\frac{13}{2}\mathrm{Li}_2(x)X-\frac{17}{6}\mathrm{Li}_2(x)\pi^2-\\
&\quad-10\mathrm{Li}_2(y)XY+2\mathrm{Li}_2(y)Y^2-6\mathrm{Li}_2(y)Y-\frac{11}{48}X^4-\frac{31}{12}X^3-5X^2Y^2+\\
&\quad+\frac{1}{4}X^2Y-\frac{5}{6}X^2\pi^2+\frac{305}{72}X^2+\frac{3}{2}XY^3-\frac{29}{6}XY^2-\frac{5}{6}XY\pi^2-\frac{37}{36}XY-\\
&\quad-13X\zeta_3-\frac{25}{3}X\pi^2+\frac{383}{36}X-\frac{5}{4}Y^2\pi^2+7Y\zeta_3+\frac{3}{2}Y\pi^2+\frac{3}{4}Y-\\
&\quad-10\zeta_3+\frac{19}{45}\pi^4-\frac{407}{72}\pi^2+\frac{139}{36}\Bigg)\frac{t}{u}+\\
&+\Bigg(-\frac{9}{2}\mathrm{Li}_4(x)-12\mathrm{Li}_4(y)-12\mathrm{Li}_4(z)+12\mathrm{Li}_3(x)X-6\mathrm{Li}_3(x)Y+\frac{7}{2}\mathrm{Li}_3(x)+\\
&\quad+12\mathrm{Li}_3(y)X-\frac{15}{4}\mathrm{Li}_2(x)X^2-\frac{7}{2}\mathrm{Li}_2(x)X-3\mathrm{Li}_2(x)\pi^2-6\mathrm{Li}_2(y)XY-\\
&\quad-\frac{5}{16}X^4+X^3Y-\frac{17}{12}X^3-3X^2Y^2-\frac{5}{2}X^2Y-X^2\pi^2+\frac{173}{24}X^2-\\
&\quad-2XY\pi^2+\frac{3}{2}XY-12X\zeta_3-\frac{77}{12}X\pi^2+\frac{305}{36}X+6Y\zeta_3+\frac{1}{3}Y\pi^2-\\
&\quad-\frac{11}{2}\zeta_3+\frac{11}{20}\pi^4-\frac{103}{36}\pi^2\Bigg)\frac{t^2}{u^2}+\\
&+\Bigg(-2\mathrm{Li}_4(x)+\mathrm{Li}_3(x)X+\frac{3}{2}\mathrm{Li}_3(x)-\frac{3}{2}\mathrm{Li}_2(x)X-\frac{1}{3}\mathrm{Li}_2(x)\pi^2+\frac{1}{8}X^4-\\
&\quad-\frac{43}{36}X^3-\frac{1}{4}X^2\pi^2+\frac{359}{72}X^2-X\zeta_3-\frac{91}{36}X\pi^2-\frac{3}{2}\zeta_3+\frac{7}{90}\pi^4\Bigg)\frac{t^3}{u^3}+\\
&+\Bigg(-\frac{3}{8}X^2+\frac{3}{4}XY-\frac{3}{8}Y^2-\frac{3}{8}\pi^2\Bigg)\frac{t}{s}+\\
&+\Bigg(\Bigg(6\mathrm{Li}_3(y)-\mathrm{Li}_2(x)X+\frac{3}{2}\mathrm{Li}_2(x)-4\mathrm{Li}_2(y)Y+\frac{1}{3}X^3-\frac{7}{2}X^2Y+\frac{37}{12}X^2+\\
&\qquad+2XY^2-\frac{29}{6}XY+\frac{2}{3}X\pi^2+3X-\frac{1}{6}Y^3+\frac{3}{2}Y^2-\\
&\qquad-\frac{5}{6}Y\pi^2-\frac{91}{36}Y-\zeta_3+\pi^2-\frac{13}{4}\Bigg)+\\
&\quad+\Bigg(14\mathrm{Li}_3(y)+\frac{5}{2}\mathrm{Li}_2(x)X-\frac{1}{2}\mathrm{Li}_2(x)-6\mathrm{Li}_2(y)Y+\frac{5}{12}X^3-2X^2Y+\\
&\qquad+\frac{5}{4}X^2+3XY^2-\frac{19}{6}XY-\frac{2}{3}X\pi^2+\frac{299}{18}X+\frac{1}{6}Y^3-\frac{11}{6}Y^2-\frac{5}{6}Y\pi^2-\\
&\qquad-\frac{37}{36}Y-6\zeta_3+\frac{4}{3}\pi^2+\frac{119}{9}\Bigg)\frac{t}{u}+\\
&\quad+\Bigg(6\mathrm{Li}_3(x)+12\mathrm{Li}_3(y)-\frac{3}{2}\mathrm{Li}_2(x)X-\frac{7}{2}\mathrm{Li}_2(x)-6\mathrm{Li}_2(y)Y+\frac{1}{4}X^3+\frac{1}{2}X^2-\\
&\qquad-5XY-\frac{3}{2}X\pi^2+\frac{235}{12}X+\frac{3}{2}Y-6\zeta_3+\frac{7}{12}\pi^2+\frac{305}{36}\Bigg)\frac{t^2}{u^2}+
\end{aligned}
\qquad \text{(A.24)}
$$

$$+\left(\mathrm{Li}_3(x)-\frac{3}{2}\mathrm{Li}_2(x)+\frac{1}{3}X^3-X^2-\frac{1}{6}X\pi^2+\frac{359}{36}X-\zeta_3+\frac{1}{4}\pi^2\right)\frac{t^3}{u^3}\Bigg)i\pi$$

$$\begin{aligned}
C^{gg,(2),[1]}_{s+-+-} = \Bigg(&-\frac{5}{2}\mathrm{Li}_4(x)+\mathrm{Li}_4(y)+\mathrm{Li}_4(z)+\mathrm{Li}_3(x)X+\mathrm{Li}_3(x)Y-\frac{3}{4}\mathrm{Li}_3(x)+3\mathrm{Li}_3(y)-\\
&-\frac{1}{4}\mathrm{Li}_2(x)X^2+\frac{3}{4}\mathrm{Li}_2(x)X-\frac{1}{6}\mathrm{Li}_2(x)\pi^2-3\mathrm{Li}_2(y)Y-\frac{1}{48}X^4-\\
&-\frac{1}{6}X^3Y-\frac{3}{8}X^3+\frac{1}{4}X^2Y^2+\frac{3}{8}X^2Y+\frac{1}{3}X^2\pi^2+\frac{5}{16}X^2-\frac{3}{2}XY^2-\\
&-\frac{1}{6}XY\pi^2-X\zeta_3+X\pi^2+\frac{39}{16}X-\frac{11}{8}Y^2-Y\zeta_3+\frac{3}{2}Y+\frac{9}{4}\zeta_3+\\
&+\frac{31}{360}\pi^4+\frac{5}{16}\pi^2-\frac{187}{64}\Bigg)+\\
+\Bigg(&-\frac{7}{2}\mathrm{Li}_4(x)+7\mathrm{Li}_4(y)+7\mathrm{Li}_4(z)-\mathrm{Li}_3(x)X+7\mathrm{Li}_3(x)Y+\frac{3}{4}\mathrm{Li}_3(x)+\\
&+9\mathrm{Li}_3(y)-\frac{3}{4}\mathrm{Li}_2(x)X^2-\frac{3}{4}\mathrm{Li}_2(x)X+\frac{1}{6}\mathrm{Li}_2(x)\pi^2-9\mathrm{Li}_2(y)Y-\frac{1}{16}X^4-\\
&-\frac{7}{6}X^3Y-\frac{31}{24}X^3+\frac{7}{4}X^2Y^2+\frac{5}{8}X^2Y+\frac{2}{3}X^2\pi^2+\frac{27}{16}X^2-\frac{9}{2}XY^2-\\
&-\frac{7}{6}XY\pi^2-\frac{5}{4}XY+X\zeta_3+2X\pi^2+\frac{97}{16}X-\frac{3}{2}Y^2-7Y\zeta_3-\frac{2}{3}Y\pi^2+\frac{3}{4}Y-\\
&-\frac{1}{2}\zeta_3+\frac{1}{90}\pi^4+\frac{65}{24}\pi^2+\frac{17}{16}\Bigg)\frac{t}{u}+\\
+\Bigg(&\frac{3}{2}\mathrm{Li}_4(x)+12\mathrm{Li}_4(y)+12\mathrm{Li}_4(z)-6\mathrm{Li}_3(x)X+12\mathrm{Li}_3(x)Y+6\mathrm{Li}_3(y)-\\
&-\frac{3}{4}\mathrm{Li}_2(x)X^2+\mathrm{Li}_2(x)\pi^2-6\mathrm{Li}_2(y)Y-\frac{1}{16}X^4-2X^3Y-\frac{11}{12}X^3+\\
&+3X^2Y^2+\frac{1}{4}X^2Y+\frac{1}{4}X^2\pi^2+\frac{35}{16}X^2-3XY^2-2XY\pi^2-\\
&-\frac{3}{2}XY+6X\zeta_3+\frac{9}{4}X\pi^2+\frac{23}{8}X-\\
&-12Y\zeta_3-\frac{2}{3}Y\pi^2+\frac{5}{2}\zeta_3-\frac{11}{60}\pi^4+\frac{23}{12}\pi^2\Bigg)\frac{t^2}{u^2}+\\
+\Bigg(&\frac{5}{2}\mathrm{Li}_4(x)+6\mathrm{Li}_4(y)+6\mathrm{Li}_4(z)-4\mathrm{Li}_3(x)X+6\mathrm{Li}_3(x)Y-\frac{3}{2}\mathrm{Li}_3(x)-\\
&-\frac{1}{4}\mathrm{Li}_2(x)X^2+\frac{3}{2}\mathrm{Li}_2(x)X+\frac{2}{3}\mathrm{Li}_2(x)\pi^2-\frac{1}{48}X^4-X^3Y+\frac{3}{2}X^2Y^2-\\
&-\frac{1}{12}X^2\pi^2+\frac{11}{16}X^2-XY\pi^2+4X\zeta_3+\frac{5}{4}X\pi^2-6Y\zeta_3+\frac{3}{2}\zeta_3-\frac{5}{36}\pi^4\Bigg)\frac{t^3}{u^3}+\\
+\Bigg(&-\frac{3}{8}X^2+\frac{3}{4}XY-\frac{3}{8}Y^2-\frac{3}{8}\pi^2\Bigg)\frac{t}{s}+\\
+\Bigg(&\Bigg(2\mathrm{Li}_3(x)-\frac{1}{2}\mathrm{Li}_2(x)X+\frac{15}{4}\mathrm{Li}_2(x)-\frac{1}{4}X^3-\frac{9}{8}X^2+\frac{3}{4}XY+\frac{5}{8}X-\\
&\quad-\frac{11}{4}Y-2\zeta_3-\frac{1}{4}\pi^2+\frac{63}{16}\Bigg)+\\
&+\Bigg(6\mathrm{Li}_3(x)-\frac{3}{2}\mathrm{Li}_2(x)X+\frac{33}{4}\mathrm{Li}_2(x)-\frac{3}{4}X^3-\frac{23}{8}X^2+\frac{5}{4}XY+\frac{17}{8}X-\\
&\quad-\frac{17}{4}Y-6\zeta_3-\frac{3}{4}\pi^2+\frac{109}{16}\Bigg)\frac{t}{u}+
\end{aligned}$$

$$
\begin{aligned}
&+\left(6\mathrm{Li}_3(x)-\frac{3}{2}\mathrm{Li}_2(x)X+6\mathrm{Li}_2(x)-\frac{3}{4}X^3-\frac{5}{2}X^2+\frac{1}{2}XY+\frac{23}{8}X-\right.\\
&\quad\left.-\frac{3}{2}Y-6\zeta_3-\frac{3}{4}\pi^2+\frac{23}{8}\right)\frac{t^2}{u^2}+ \qquad\qquad (A.25)\\
&+\left(2\mathrm{Li}_3(x)-\frac{1}{2}\mathrm{Li}_2(x)X+\frac{3}{2}\mathrm{Li}_2(x)-\frac{1}{4}X^3-\frac{3}{4}X^2+\frac{11}{8}X-2\zeta_3-\frac{1}{4}\pi^2\right)\frac{t^3}{u^3}\Bigg)i\pi
\end{aligned}
$$

$$
\begin{aligned}
D^{gg,(2),[1]}_{s+-+-} = &\left(-\frac{2}{3}\mathrm{Li}_3(x)+\frac{2}{3}\mathrm{Li}_2(x)X-\frac{7}{36}X^3+\frac{1}{3}X^2Y+\frac{43}{36}X^2-\frac{13}{36}X\pi^2-\right.\\
&\left.-\frac{5}{9}X+\frac{17}{12}\zeta_3-\frac{77}{108}\pi^2-\frac{1307}{432}\right)+\\
&+\left(-\frac{1}{2}X^2-\frac{19}{18}X-\frac{5}{36}\pi^2-\frac{37}{36}\right)\frac{t}{u}+\left(-\frac{3}{4}X^2-\frac{37}{18}X+\frac{5}{18}\pi^2\right)\frac{t^2}{u^2}+\\
&+\left(\frac{7}{36}X^3-\frac{37}{36}X^2+\frac{5}{18}X\pi^2\right)\frac{t^3}{u^3}+\\
&+\Bigg(\left(\frac{2}{3}\mathrm{Li}_2(x)-\frac{1}{4}X^2+\frac{2}{3}XY-\frac{1}{18}X-\frac{1}{12}\pi^2-\frac{13}{18}\right)+\\
&\quad+\left(-\frac{2}{3}X-\frac{25}{18}\right)\frac{t}{u}+\left(-\frac{13}{6}X-\frac{37}{18}\right)\frac{t^2}{u^2}+\left(\frac{1}{4}X^2-\frac{37}{18}X\right)\frac{t^3}{u^3}\Bigg)i\pi \qquad (A.26)
\end{aligned}
$$

$$
\begin{aligned}
E^{gg,(2),[1]}_{s+-+-} = &\left(-\frac{2}{3}\mathrm{Li}_3(y)+\frac{2}{3}\mathrm{Li}_2(y)Y+\frac{7}{36}X^3-\frac{1}{3}X^2Y+\frac{2}{9}X^2+\frac{1}{2}XY^2-\frac{5}{9}XY+\frac{4}{9}X\pi^2-\right.\\
&\left.-\frac{11}{6}X-\frac{1}{6}Y^3-\frac{17}{18}Y^2+\frac{7}{18}Y\pi^2+3Y+\frac{23}{36}\zeta_3+\frac{37}{72}\pi^2+\frac{3401}{1296}\right)+\\
&+\left(-4\mathrm{Li}_4(x)+4\mathrm{Li}_4(y)+4\mathrm{Li}_4(z)+4\mathrm{Li}_3(x)Y+8\mathrm{Li}_3(y)-\frac{2}{3}\mathrm{Li}_2(x)\pi^2-\right.\\
&\quad-8\mathrm{Li}_2(y)Y+\frac{1}{6}X^4-\frac{2}{3}X^3Y+\frac{1}{2}X^3+X^2Y^2-\frac{1}{4}X^2Y+\frac{1}{3}X^2\pi^2+\frac{1}{9}X^2-\\
&\quad-\frac{11}{3}XY^2-\frac{2}{3}XY\pi^2+\frac{8}{9}XY+X\pi^2+\frac{7}{9}X-2Y^2-4Y\zeta_3+\\
&\quad\left.+\frac{7}{6}Y\pi^2+\frac{5}{3}Y-8\zeta_3+\frac{7}{45}\pi^4-\frac{7}{12}\pi^2-\frac{19}{36}\right)\frac{t}{u}+\\
&+\left(-12\mathrm{Li}_4(x)+12\mathrm{Li}_4(y)+12\mathrm{Li}_4(z)+12\mathrm{Li}_3(x)Y+8\mathrm{Li}_3(y)-2\mathrm{Li}_2(x)\pi^2-\right.\\
&\quad-8\mathrm{Li}_2(y)Y+\frac{1}{2}X^4-2X^3Y+\frac{1}{3}X^3+3X^2Y^2+\frac{1}{4}X^2Y+X^2\pi^2+\\
&\quad+\frac{11}{12}X^2-4XY^2-2XY\pi^2-\frac{2}{3}XY+\frac{5}{6}X\pi^2+\frac{11}{18}X-12Y\zeta_3+\frac{4}{3}Y\pi^2-\\
&\quad\left.-8\zeta_3+\frac{7}{15}\pi^4-\frac{1}{2}\pi^2\right)\frac{t^2}{u^2}+\\
&+\left(-8\mathrm{Li}_4(x)+8\mathrm{Li}_4(y)+8\mathrm{Li}_4(z)+8\mathrm{Li}_3(x)Y-\frac{2}{3}\mathrm{Li}_3(x)+\frac{2}{3}\mathrm{Li}_2(x)X-\right.\\
&\quad-\frac{4}{3}\mathrm{Li}_2(x)\pi^2+\frac{1}{3}X^4-\frac{4}{3}X^3Y+\frac{7}{36}X^3+2X^2Y^2+\frac{2}{3}X^2\pi^2-\frac{1}{36}X^2-\\
&\quad\left.-\frac{4}{3}XY\pi^2+\frac{5}{6}X\pi^2-8Y\zeta_3+\frac{2}{3}\zeta_3+\frac{14}{45}\pi^4\right)\frac{t^3}{u^3}-\frac{3}{2}Y^2\frac{s}{t}+
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\bigg(-\frac{2}{3}\mathrm{Li}_2(x)-\frac{1}{12}X^2-\frac{1}{3}XY-\frac{10}{9}X-\frac{1}{3}Y^2-\frac{22}{9}Y+3\bigg)+\\
&\quad+\bigg(4\mathrm{Li}_3(x)+8\mathrm{Li}_2(x)+\frac{1}{4}X^2+\frac{1}{6}XY-\frac{5}{9}X+\frac{1}{3}Y^2-\frac{28}{9}Y-4\zeta_3+\frac{19}{9}\bigg)\frac{t}{u}+\\
&\quad+\bigg(12\mathrm{Li}_3(x)+8\mathrm{Li}_2(x)+\frac{1}{4}X^2+\frac{1}{2}XY+\frac{1}{2}X-\frac{2}{3}Y-12\zeta_3+\frac{11}{18}\bigg)\frac{t^2}{u^2}+\\
&\quad+\bigg(8\mathrm{Li}_3(x)+\frac{2}{3}\mathrm{Li}_2(x)-\frac{1}{12}X^2-\frac{1}{18}X-8\zeta_3-\frac{1}{9}\pi^2\bigg)\frac{t^3}{u^3}-3Y\frac{s}{t}\Bigg)i\pi
\end{aligned}
\tag{A.27}
$$

$$
F^{gg,(2),[1]}_{s+-+-}=\bigg(\frac{1}{18}X+\frac{5}{27}\bigg)i\pi+\bigg(-\frac{1}{12}X^2+\frac{5}{27}X+\frac{2}{27}\pi^2\bigg)
\tag{A.28}
$$

$$
\begin{aligned}
G^{gg,(2),[1]}_{s+-+-}=&\bigg(\frac{1}{36}XY-\frac{1}{36}Y^2\bigg)+\bigg(\frac{1}{36}X^2-\frac{1}{36}XY\bigg)\frac{t}{u}+\\
&+\Bigg(\bigg(\frac{1}{36}X-\frac{1}{36}Y\bigg)+\bigg(\frac{1}{36}X-\frac{1}{36}Y\bigg)\frac{t}{u}\Bigg)i\pi
\end{aligned}
\tag{A.29}
$$

$$
\begin{aligned}
A^{gg,(2),[2]}_{s+-+-}=&\bigg(-2\mathrm{Li}_4(x)+\mathrm{Li}_4(y)+2\mathrm{Li}_4(z)+2\mathrm{Li}_3(x)Y+\mathrm{Li}_3(y)Y+\frac{41}{3}\mathrm{Li}_3(y)-\\
&-\frac{2}{3}\mathrm{Li}_2(x)\pi^2-\frac{1}{2}\mathrm{Li}_2(y)Y^2-\frac{41}{3}\mathrm{Li}_2(y)Y+\frac{1}{12}X^4-\frac{1}{3}X^3Y+\frac{1}{2}X^2Y^2+\\
&+\frac{1}{6}X^2\pi^2-\frac{1}{6}XY^3-\frac{41}{6}XY^2-\frac{1}{3}XY\pi^2-\frac{5}{24}Y^4+\frac{26}{9}Y^3+Y^2\pi^2-\frac{145}{18}Y^2-\\
&-Y\zeta_3+\frac{247}{72}Y\pi^2-\frac{2}{27}Y-\frac{445}{24}\zeta_3+\frac{121}{1440}\pi^4-\frac{13}{6}\pi^2+\frac{30593}{1728}\bigg)+\\
&+\bigg(-12\mathrm{Li}_4(x)+12\mathrm{Li}_4(y)+12\mathrm{Li}_4(z)+12\mathrm{Li}_3(x)Y+10\mathrm{Li}_3(y)-2\mathrm{Li}_2(x)\pi^2-\\
&\quad-10\mathrm{Li}_2(y)Y+\frac{1}{2}X^4-2X^3Y+3X^2Y^2+X^2\pi^2-5XY^2-2XY\pi^2-\\
&\quad-12Y\zeta_3+\frac{5}{3}Y\pi^2-10\zeta_3+\frac{7}{15}\pi^4-\frac{5}{3}\pi^2+\frac{1}{8}\bigg)\frac{t}{u}+\\
&+\bigg(-10\mathrm{Li}_4(x)+10\mathrm{Li}_4(y)+10\mathrm{Li}_4(z)+10\mathrm{Li}_3(x)Y-\frac{5}{3}\mathrm{Li}_2(x)\pi^2+\frac{5}{12}X^4-\\
&\quad-\frac{5}{3}X^3Y+\frac{5}{2}X^2Y^2+\frac{5}{6}X^2\pi^2-\frac{5}{3}XY\pi^2-10Y\zeta_3+\frac{7}{18}\pi^4\bigg)\frac{t^2}{u^2}+\\
&+\bigg(-\frac{3}{2}\mathrm{Li}_4(y)+\mathrm{Li}_3(y)Y+3\mathrm{Li}_3(y)-\frac{1}{4}\mathrm{Li}_2(y)Y^2-3\mathrm{Li}_2(y)Y-\frac{3}{2}XY^2-\\
&\quad-\frac{7}{48}Y^4+\frac{13}{9}Y^3+\frac{2}{3}Y^2\pi^2+\frac{713}{144}Y^2-Y\zeta_3+\frac{23}{18}Y\pi^2-\\
&\quad-5Y-3\zeta_3+\frac{1}{60}\pi^4\bigg)\frac{s}{t}+\frac{5}{2}Y^2\frac{s^2}{t^2}+\\
&+\Bigg(\bigg(2\mathrm{Li}_3(x)+\mathrm{Li}_3(y)+\frac{41}{3}\mathrm{Li}_2(x)-\mathrm{Li}_2(y)Y-\frac{1}{2}XY^2-\frac{5}{6}Y^3+\\
&\qquad+5Y^2+\frac{1}{3}Y\pi^2-\frac{35}{9}Y-\zeta_3-\frac{29}{72}\pi^2+\frac{142}{27}\bigg)+\\
&+\bigg(12\mathrm{Li}_3(x)+10\mathrm{Li}_2(x)-12\zeta_3\bigg)\frac{t}{u}+5Y\frac{s^2}{t^2}+\bigg(10\mathrm{Li}_3(x)-10\zeta_3\bigg)\frac{t^2}{u^2}+
\end{aligned}
\tag{A.30}
$$

$$+\left(\mathrm{Li}_3(y)+3\mathrm{Li}_2(x)-\frac{1}{2}\mathrm{Li}_2(y)Y-\frac{7}{12}Y^3+\frac{5}{2}Y^2+\frac{1}{6}Y\pi^2+\frac{713}{72}Y-\zeta_3-5\right)\frac{s}{t}\Bigg)i\pi$$

$$\begin{aligned}
B^{gg,(2),[2]}_{s+-+-} = \Bigg(&-6\mathrm{Li}_4(x)+15\mathrm{Li}_4(y)+20\mathrm{Li}_4(z)-8\mathrm{Li}_3(x)X+14\mathrm{Li}_3(x)Y+\mathrm{Li}_3(x)-\\
&-12\mathrm{Li}_3(y)X+5\mathrm{Li}_3(y)Y+7\mathrm{Li}_3(y)+\mathrm{Li}_2(x)X^2-\mathrm{Li}_2(x)X+\frac{7}{3}\mathrm{Li}_2(x)\pi^2+\\
&+6\mathrm{Li}_2(y)XY-\frac{5}{2}\mathrm{Li}_2(y)Y^2-7\mathrm{Li}_2(y)Y+\frac{3}{4}X^4-3X^3Y+\frac{1}{6}X^3+\\
&+\frac{11}{2}X^2Y^2+\frac{1}{3}X^2Y+\frac{5}{12}X^2\pi^2-\frac{41}{18}X^2-\frac{5}{6}XY^3-\frac{31}{6}XY^2-\frac{4}{3}XY\pi^2+\\
&+\frac{19}{18}XY+12X\zeta_3-\frac{5}{6}X\pi^2+\frac{3}{2}X-\frac{1}{24}Y^4+\frac{5}{3}Y^3+\frac{5}{6}Y^2\pi^2-\frac{145}{36}Y^2-\\
&-14Y\zeta_3-\frac{5}{6}Y\pi^2+\frac{71}{12}Y+\frac{731}{72}\zeta_3-\frac{383}{1440}\pi^4+\frac{343}{144}\pi^2-\frac{18815}{1296}\Bigg)+\\
&+\Bigg(3\mathrm{Li}_4(x)+27\mathrm{Li}_4(y)+27\mathrm{Li}_4(z)-18\mathrm{Li}_3(x)X+21\mathrm{Li}_3(x)Y-\frac{5}{2}\mathrm{Li}_3(x)\\
&\quad-12\mathrm{Li}_3(y)X+7\mathrm{Li}_3(y)+3\mathrm{Li}_2(x)X^2+\frac{5}{2}\mathrm{Li}_2(x)X+\frac{7}{2}\mathrm{Li}_2(x)\pi^2+\\
&\quad+6\mathrm{Li}_2(y)XY-7\mathrm{Li}_2(y)Y+\frac{7}{8}X^4-\frac{7}{2}X^3Y-\frac{1}{2}X^3+\frac{27}{4}X^2Y^2+\frac{9}{4}X^2Y-\\
&\quad-X^2\pi^2-\frac{13}{8}X^2-\frac{7}{2}XY^2-\frac{1}{2}XY\pi^2-3XY+18X\zeta_3-\frac{5}{12}X\pi^2+\frac{3}{4}X-\\
&\quad-21Y\zeta_3-\frac{5}{6}Y\pi^2+\frac{15}{2}\zeta_3-\frac{37}{60}\pi^4+\frac{5}{4}\pi^2+\frac{1}{4}\Bigg)\frac{t}{u}+\\
&+\Bigg(5\mathrm{Li}_4(x)+7\mathrm{Li}_4(y)+7\mathrm{Li}_4(z)-6\mathrm{Li}_3(x)X+7\mathrm{Li}_3(x)Y-\frac{7}{2}\mathrm{Li}_3(x)+\\
&\quad+\frac{7}{2}\mathrm{Li}_2(x)X+\frac{5}{6}\mathrm{Li}_2(x)\pi^2+\frac{1}{24}X^4-\frac{7}{6}X^3Y+X^3+\frac{7}{4}X^2Y^2+\frac{1}{4}X^2Y-\\
&\quad-\frac{5}{12}X^2\pi^2-\frac{11}{8}X^2-\frac{7}{6}XY\pi^2+6X\zeta_3+\frac{5}{12}X\pi^2-\\
&\quad-7Y\zeta_3+\frac{7}{2}\zeta_3-\frac{7}{36}\pi^4\Bigg)\frac{t^2}{u^2}+\\
&+\left(\frac{3}{8}X^2-\frac{3}{4}XY+\frac{3}{8}Y^2+\frac{3}{8}\pi^2\right)\frac{t}{s}+\\
&+\Bigg(3\mathrm{Li}_4(x)-7\mathrm{Li}_4(y)+\mathrm{Li}_4(z)-2\mathrm{Li}_3(x)Y+6\mathrm{Li}_3(x)+\mathrm{Li}_3(y)X+4\mathrm{Li}_3(y)Y+\\
&\quad+3\mathrm{Li}_3(y)-2\mathrm{Li}_2(x)X^2-6\mathrm{Li}_2(x)X-\frac{1}{6}\mathrm{Li}_2(x)\pi^2-4\mathrm{Li}_2(y)XY-3\mathrm{Li}_2(y)Y+\\
&\quad+\frac{1}{24}X^4-\frac{5}{3}X^3Y-\frac{7}{4}X^2Y^2+\frac{1}{3}X^2Y+\frac{4}{3}X^2\pi^2+\frac{5}{6}XY^3-5XY^2-\\
&\quad-\frac{5}{6}XY\pi^2+\frac{25}{9}XY-X\zeta_3-\frac{5}{3}X\pi^2-\frac{1}{6}Y^4+\frac{28}{9}Y^3+\frac{1}{6}Y^2\pi^2-\frac{199}{72}Y^2+\\
&\quad+2Y\zeta_3+\frac{35}{18}Y\pi^2-2Y-3\zeta_3+\frac{7}{72}\pi^4\Bigg)\frac{s}{t}+Y^2\frac{s^2}{t^2}+\\
&+\Bigg(\Bigg(6\mathrm{Li}_3(x)-7\mathrm{Li}_3(y)-4\mathrm{Li}_2(x)X+6\mathrm{Li}_2(x)+\mathrm{Li}_2(y)Y+\frac{2}{3}X^3-X^2Y+\\
&\qquad+\frac{4}{3}X^2-\frac{3}{2}XY^2-\frac{8}{3}XY+\frac{1}{2}X\pi^2-\frac{7}{2}X-\frac{1}{6}Y^3+\frac{10}{3}Y^2-7Y-
\end{aligned}$$

$$
\begin{aligned}
&-2\zeta_3-\frac{5}{6}\pi^2-\frac{29}{4}\Bigg)+\\
&+\Bigg(3\mathrm{Li}_3(x)-12\mathrm{Li}_3(y)+\frac{19}{2}\mathrm{Li}_2(x)+6\mathrm{Li}_2(y)Y+X^3-\frac{1}{2}X^2+\frac{9}{2}XY+\pi^2\\
&\quad+\frac{1}{2}X-\frac{25}{4}X-3Y-3\zeta_3-\frac{17}{12}\pi^2+\frac{3}{4}\Bigg)\frac{t}{u}+\\
&+\Bigg(\mathrm{Li}_3(x)+\frac{7}{2}\mathrm{Li}_2(x)+\frac{3}{2}X^2+\frac{1}{2}XY-\frac{11}{4}X-\zeta_3-\frac{7}{12}\pi^2\Bigg)\frac{t^2}{u^2}+\\
&+\Bigg(-2\mathrm{Li}_3(x)+5\mathrm{Li}_3(y)-3\mathrm{Li}_2(x)-4\mathrm{Li}_2(y)Y-\frac{1}{6}X^3-3X^2Y+\frac{10}{3}X^2+\\
&\quad+2XY^2-\frac{19}{3}XY+\frac{5}{6}X\pi^2+\frac{25}{9}X-\frac{1}{2}Y^3+4Y^2-\frac{2}{3}Y\pi^2-\frac{11}{4}Y+\\
&\quad+\zeta_3+2\pi^2-2\Bigg)\frac{s}{t}+2Y\frac{s^2}{t^2}\Bigg)i\pi
\end{aligned}
\tag{A.31}
$$

$$
\begin{aligned}
C^{gg,(2),[2]}_{s+-+-}=&\Bigg(-2\mathrm{Li}_4(x)+2\mathrm{Li}_4(y)+2\mathrm{Li}_4(z)+2\mathrm{Li}_3(x)Y+\mathrm{Li}_3(x)+\mathrm{Li}_3(y)-\mathrm{Li}_2(x)X-\\
&-\frac{1}{3}\mathrm{Li}_2(x)\pi^2-\mathrm{Li}_2(y)Y+\frac{1}{12}X^4-\frac{1}{3}X^3Y+\frac{1}{2}X^2Y^2-\frac{1}{2}X^2Y+\frac{1}{6}X^2\pi^2+\\
&+\frac{1}{2}X^2-\frac{1}{2}XY^2-\frac{1}{3}XY\pi^2-\frac{3}{2}XY+\frac{1}{6}X\pi^2+\frac{3}{2}X+\frac{3}{4}Y^2-2Y\zeta_3+\frac{1}{6}Y\pi^2-\\
&-\frac{3}{4}Y+\frac{7}{4}\zeta_3+\frac{13}{120}\pi^4-\frac{5}{48}\pi^2-\frac{247}{64}\Bigg)+\\
&+\Bigg(-3\mathrm{Li}_4(x)+3\mathrm{Li}_4(y)+3\mathrm{Li}_4(z)+3\mathrm{Li}_3(x)Y+\frac{1}{2}\mathrm{Li}_3(x)+\mathrm{Li}_3(y)-\\
&\quad-\frac{1}{2}\mathrm{Li}_2(x)X-\frac{1}{2}\mathrm{Li}_2(x)\pi^2-\mathrm{Li}_2(y)Y+\frac{1}{8}X^4-\frac{1}{2}X^3Y+\frac{3}{4}X^2Y^2-\frac{1}{4}X^2Y+\\
&\quad+\frac{1}{4}X^2\pi^2-\frac{1}{8}X^2-\frac{1}{2}XY^2-\frac{1}{2}XY\pi^2+\frac{1}{12}X\pi^2+\frac{3}{4}X-3Y\zeta_3\\
&\quad+\frac{1}{6}Y\pi^2-\frac{3}{2}\zeta_3+\frac{7}{60}\pi^4-\frac{1}{4}\pi^2+\frac{1}{8}\Bigg)\frac{t}{u}+\\
&+\Bigg(-\mathrm{Li}_4(x)+\mathrm{Li}_4(y)+\mathrm{Li}_4(z)+\mathrm{Li}_3(x)Y-\frac{1}{2}\mathrm{Li}_3(x)+\frac{1}{2}\mathrm{Li}_2(x)X-\frac{1}{6}\mathrm{Li}_2(x)\pi^2+\\
&\quad+\frac{1}{24}X^4-\frac{1}{6}X^3Y+\frac{1}{4}X^2Y^2+\frac{1}{4}X^2Y+\frac{1}{12}X^2\pi^2+\frac{1}{8}X^2-\\
&\quad-\frac{1}{6}XY\pi^2-\frac{1}{12}X\pi^2-Y\zeta_3+\frac{1}{2}\zeta_3+\frac{7}{180}\pi^4\Bigg)\frac{t^2}{u^2}+\\
&+\Bigg(\frac{3}{8}X^2-\frac{3}{4}XY+\frac{3}{8}Y^2+\frac{3}{8}\pi^2\Bigg)\frac{t}{s}+\\
&+\Bigg(-\mathrm{Li}_4(x)+\frac{5}{2}\mathrm{Li}_4(y)+\mathrm{Li}_4(z)-\mathrm{Li}_3(y)X-\mathrm{Li}_3(y)Y-3\mathrm{Li}_3(y)-\frac{1}{6}\mathrm{Li}_2(x)\pi^2+\\
&\quad+\frac{1}{4}\mathrm{Li}_2(y)Y^2+3\mathrm{Li}_2(y)Y+\frac{1}{24}X^4-\frac{1}{6}X^3Y+\frac{1}{12}X^2\pi^2+\frac{3}{2}XY^2-\\
&\quad-\frac{1}{6}XY\pi^2+X\zeta_3+\frac{1}{16}Y^4-\frac{1}{4}Y^2\pi^2-\frac{15}{16}Y^2+Y\zeta_3-\\
&\quad-\frac{1}{2}Y\pi^2+Y+3\zeta_3-\frac{1}{120}\pi^4\Bigg)\frac{s}{t}-\frac{1}{2}Y^2\frac{s^2}{t^2}+
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\bigg(2\mathrm{Li}_3(x) - XY - \frac{1}{2}X - 2\zeta_3 + \frac{1}{6}\pi^2 + \frac{3}{4}\bigg)+\\
&\quad+\bigg(3\mathrm{Li}_3(x) + \frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}XY - \frac{1}{4}X - 3\zeta_3 + \frac{1}{12}\pi^2 + \frac{3}{4}\bigg)\frac{t}{u}+\\
&\quad+\bigg(\mathrm{Li}_3(x) + \frac{1}{2}\mathrm{Li}_2(x) + \frac{1}{2}XY + \frac{1}{4}X - \zeta_3 - \frac{1}{12}\pi^2\bigg)\frac{t^2}{u^2}+\\
&\quad+\bigg(-2\mathrm{Li}_3(y) - 3\mathrm{Li}_2(x) + \frac{1}{2}\mathrm{Li}_2(y)Y + \frac{1}{4}Y^3 - \frac{15}{8}Y + 2\zeta_3 + 1\bigg)\frac{s}{t} - - Y\frac{s^2}{t^2}\Bigg)i\pi
\end{aligned}
\tag{A.32}
$$

$$
\begin{aligned}
D^{gg,(2),[2]}_{s+-+-} = &\bigg(2\mathrm{Li}_4(x) - 2\mathrm{Li}_4(y) - 2\mathrm{Li}_4(z) - 2\mathrm{Li}_3(x)Y - \frac{14}{3}\mathrm{Li}_3(y) + \frac{1}{3}\mathrm{Li}_2(x)\pi^2+\\
&+\frac{14}{3}\mathrm{Li}_2(y)Y - \frac{1}{12}X^4 + \frac{1}{3}X^3Y - \frac{1}{2}X^2Y^2 - \frac{1}{6}X^2\pi^2 + \frac{7}{3}XY^2 + \frac{1}{3}XY\pi^2-\\
&-\frac{7}{18}Y^3 + \frac{71}{36}Y^2 + 2Y\zeta_3 - \frac{47}{36}Y\pi^2 - \frac{14}{9}Y + \frac{65}{12}\zeta_3 - \frac{7}{90}\pi^4 + \frac{10}{27}\pi^2 - \frac{863}{432}\bigg)+\\
&+\bigg(6\mathrm{Li}_4(x) - 6\mathrm{Li}_4(y) - 6\mathrm{Li}_4(z) - 6\mathrm{Li}_3(x)Y - 4\mathrm{Li}_3(y) + \mathrm{Li}_2(x)\pi^2+\\
&\quad+4\mathrm{Li}_2(y)Y - \frac{1}{4}X^4 + X^3Y - \frac{3}{2}X^2Y^2 - \frac{1}{2}X^2\pi^2 + 2XY^2 + XY\pi^2 + 6Y\zeta_3-\\
&\quad-\frac{2}{3}Y\pi^2 + 4\zeta_3 - \frac{7}{30}\pi^4 + \frac{2}{3}\pi^2\bigg)\frac{t}{u}+\\
&+\bigg(4\mathrm{Li}_4(x) - 4\mathrm{Li}_4(y) - 4\mathrm{Li}_4(z) - 4\mathrm{Li}_3(x)Y + \frac{2}{3}\mathrm{Li}_2(x)\pi^2 - \frac{1}{6}X^4+\\
&\quad+\frac{2}{3}X^3Y - X^2Y^2 - \frac{1}{3}X^2\pi^2 + \frac{2}{3}XY\pi^2 + 4Y\zeta_3 - \frac{7}{45}\pi^4\bigg)\frac{t^2}{u^2}+\\
&+\bigg(-\frac{7}{36}Y^3 - \frac{71}{36}Y^2 - \frac{5}{18}Y\pi^2 + 2Y\bigg)\frac{s}{t} - Y^2\frac{s^2}{t^2}\\
&+\Bigg(\bigg(-2\mathrm{Li}_3(x) - \frac{14}{3}\mathrm{Li}_2(x) - \frac{1}{2}Y^2 + \frac{1}{2}Y + 2\zeta_3 + \frac{1}{36}\pi^2 - \frac{25}{18}\bigg)+\\
&\quad+\bigg(-6\mathrm{Li}_3(x) - 4\mathrm{Li}_2(x) + 6\zeta_3\bigg)\frac{t}{u} + \bigg(-4\mathrm{Li}_3(x) + 4\zeta_3\bigg)\frac{t^2}{u^2}+\\
&\quad+\bigg(-\frac{1}{4}Y^2 - \frac{71}{18}Y + 2\bigg)\frac{s}{t} - 2Y\frac{s^2}{t^2}\Bigg)i\pi
\end{aligned}
\tag{A.33}
$$

$$
\begin{aligned}
E^{gg,(2),[2]}_{s+-+-} = &\bigg(-4\mathrm{Li}_4(x) + 4\mathrm{Li}_4(y) + 4\mathrm{Li}_4(z) + 4\mathrm{Li}_3(x)Y + \frac{26}{3}\mathrm{Li}_3(y) - \frac{2}{3}\mathrm{Li}_2(x)\pi^2-\\
&-\frac{26}{3}\mathrm{Li}_2(y)Y + \frac{1}{6}X^4 - \frac{2}{3}X^3Y + \frac{1}{12}X^3 + X^2Y^2 - \frac{1}{12}X^2Y + \frac{1}{3}X^2\pi^2+\\
&+\frac{14}{9}X^2 - \frac{23}{6}XY^2 - \frac{2}{3}XY\pi^2 - \frac{7}{9}XY + 3X - \frac{1}{6}Y^3 + \frac{1}{18}Y^2-\\
&-4Y\zeta_3 + \frac{7}{9}Y\pi^2 - 2Y - \frac{313}{36}\zeta_3 + \frac{7}{45}\pi^4 - \frac{85}{72}\pi^2 + \frac{4085}{1296}\bigg)+\\
&+\bigg(-12\mathrm{Li}_4(x) + 12\mathrm{Li}_4(y) + 12\mathrm{Li}_4(z) + 12\mathrm{Li}_3(x)Y + 8\mathrm{Li}_3(y)-\\
&\quad-2\mathrm{Li}_2(x)\pi^2 8\mathrm{Li}_2(y)Y + \frac{1}{2}X^4 - 2X^3Y - +\frac{1}{4}X^3 + 3X^2Y^2-
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{4}X^2Y+X^2\pi^2+2X^2-4XY^2--2XY\pi^2-\frac{2}{3}XY+\frac{5}{3}X- \\
& -12Y\zeta_3+\frac{4}{3}Y\pi^2-8\zeta_3+\frac{7}{15}\pi^4-\frac{7}{9}\pi^2\Big)\frac{t}{u}+ \\
& +\Big(-8\mathrm{Li}_4(x)+8\mathrm{Li}_4(y)+8\mathrm{Li}_4(z)+8\mathrm{Li}_3(x)Y-\frac{2}{3}\mathrm{Li}_3(x)+\frac{2}{3}\mathrm{Li}_2(x)X- \\
& -\frac{4}{3}\mathrm{Li}_2(x)\pi^2+\frac{1}{3}X^4-\frac{4}{3}X^3Y+2X^2Y^2+\frac{2}{3}X^2\pi^2+\frac{1}{2}X^2- \\
& -\frac{4}{3}XY\pi^2+\frac{5}{9}X\pi^2-8Y\zeta_3+\frac{2}{3}\zeta_3+\frac{14}{45}\pi^4\Big)\frac{t^2}{u^2}+ \\
& +\Big(\frac{2}{3}\mathrm{Li}_3(y)-\frac{2}{3}\mathrm{Li}_2(y)Y-\frac{1}{3}X^2Y+\frac{1}{2}XY^2-\frac{5}{9}XY+\frac{1}{6}X\pi^2- \\
& -\frac{13}{36}Y^3+\frac{49}{12}Y^2-Y\pi^2-3Y-\frac{2}{3}\zeta_3\Big)\frac{s}{t}+\frac{3}{2}Y^2\frac{s^2}{t^2}+ \\
& +\Bigg(\Big(4\mathrm{Li}_3(x)+\frac{26}{3}\mathrm{Li}_2(x)+\frac{1}{6}X^2+\frac{5}{6}XY+\frac{7}{3}X-\frac{2}{3}Y-4\zeta_3+\frac{19}{6}\Big)+ \\
& +\Big(12\mathrm{Li}_3(x)+8\mathrm{Li}_2(x)+\frac{1}{2}X^2-\frac{1}{2}XY+\frac{10}{3}X-\frac{2}{3}Y-12\zeta_3+\frac{5}{3}\Big)\frac{t}{u}+ \\
& +\Big(8\mathrm{Li}_3(x)+\frac{2}{3}\mathrm{Li}_2(x)-\frac{1}{3}X^2+X-8\zeta_3-\frac{1}{9}\pi^2\Big)\frac{t^2}{u^2}+ \\
& +\Big(\frac{2}{3}\mathrm{Li}_2(x)-\frac{1}{3}X^2+XY-\frac{5}{9}X+\frac{1}{12}Y^2+\frac{137}{18}Y-3\Big)\frac{s}{t}+3Y\frac{s^2}{t^2}\Bigg)i\pi \qquad \text{(A.34)}
\end{aligned}
$$

$$
F^{gg,(2),[2]}_{s+-+-} = \Big(\frac{1}{18}Y+\frac{5}{27}\Big)i\pi+\Big(-\frac{1}{12}Y^2+\frac{5}{27}Y+\frac{2}{27}\pi^2\Big) \qquad \text{(A.35)}
$$

$$
\begin{aligned}
G^{gg,(2),[2]}_{s+-+-} = & \Big(-\frac{1}{36}X^2+\frac{1}{18}XY-\frac{1}{36}Y^2\Big)+\Big(\frac{1}{36}X-\frac{1}{36}Y\Big)i\pi\frac{s}{t}+ \\
& +\Big(\frac{1}{36}XY-\frac{1}{36}Y^2\Big)\frac{s}{t} \qquad \text{(A.36)}
\end{aligned}
$$

$$
\begin{aligned}
H^{gg,(2),[3]}_{s+-+-} = & \Big(-4\mathrm{Li}_4(x)-\frac{13}{2}\mathrm{Li}_4(y)+4\mathrm{Li}_4(z)+\mathrm{Li}_3(x)X+\frac{7}{2}\mathrm{Li}_3(x)Y-\frac{13}{6}\mathrm{Li}_3(x)- \\
& -\frac{5}{2}\mathrm{Li}_3(y)X+\frac{15}{2}\mathrm{Li}_3(y)Y+\frac{9}{4}\mathrm{Li}_3(y)-\mathrm{Li}_2(x)X^2+\frac{13}{6}\mathrm{Li}_2(x)X+ \\
& +\frac{11}{6}\mathrm{Li}_2(x)\pi^2+2\mathrm{Li}_2(y)XY-\frac{9}{4}\mathrm{Li}_2(y)Y^2-\frac{9}{4}\mathrm{Li}_2(y)Y+\frac{1}{6}X^4-X^3Y+ \\
& +\frac{7}{4}X^2Y^2+\frac{4}{3}X^2Y-\frac{1}{6}X^2\pi^2-\frac{1}{4}XY^3-\frac{23}{12}XY^2+\frac{11}{6}XY+ \\
& +\frac{5}{2}X\zeta_3-\frac{91}{24}X\pi^2-\frac{1}{16}Y^4+\frac{11}{18}Y^3-\frac{1}{4}Y^2\pi^2+\frac{161}{72}Y^2-3Y\zeta_3+ \\
& +\frac{43}{48}Y\pi^2-\frac{211}{108}Y-\frac{9}{4}\zeta_3+\frac{43}{120}\pi^4-\frac{391}{72}\pi^2\Big)+ \\
& +\Big(\frac{25}{4}\mathrm{Li}_4(x)-\frac{7}{2}\mathrm{Li}_3(x)X-\mathrm{Li}_3(x)Y-\frac{13}{3}\mathrm{Li}_3(x)-2\mathrm{Li}_3(y)X- \\
& -\frac{29}{2}\mathrm{Li}_3(y)+\frac{3}{8}\mathrm{Li}_2(x)X^2+\frac{13}{3}\mathrm{Li}_2(x)X+\frac{5}{3}\mathrm{Li}_2(x)\pi^2+\mathrm{Li}_2(y)XY+ \\
& +\frac{29}{2}\mathrm{Li}_2(y)Y+\frac{3}{32}X^4+\frac{1}{6}X^3Y-\frac{11}{12}X^3+\frac{7}{6}X^2Y-\frac{13}{8}X^2\pi^2+\frac{25}{24}X^2+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{29}{4}XY^2+\frac{2}{3}XY\pi^2+\frac{17}{4}XY+\frac{7}{2}X\zeta_3-\frac{23}{9}X\pi^2-\\
&-\frac{5}{6}X+Y\zeta_3-\frac{25}{12}Y\pi^2+\frac{46}{3}\zeta_3-\frac{25}{72}\pi^4-\frac{77}{72}\pi^2\Bigg)\frac{t}{u}+\\
&+\Bigg(11\mathrm{Li}_4(x)-\frac{33}{2}\mathrm{Li}_4(y)-\frac{33}{2}\mathrm{Li}_4(z)+\frac{7}{2}\mathrm{Li}_3(x)X-\frac{31}{2}\mathrm{Li}_3(x)Y+\frac{31}{12}\mathrm{Li}_3(x)+\\
&+2\mathrm{Li}_3(y)X-\frac{3}{4}\mathrm{Li}_2(x)X^2-\frac{31}{12}\mathrm{Li}_2(x)X+\frac{19}{12}\mathrm{Li}_2(x)\pi^2-\mathrm{Li}_2(y)XY-\\
&-\frac{11}{16}X^4+\frac{31}{12}X^3Y-\frac{1}{12}X^3-\frac{33}{8}X^2Y^2+\frac{1}{3}X^2Y-\frac{7}{8}X^2\pi^2+\frac{15}{8}X^2+\\
&+\frac{25}{12}XY\pi^2-\frac{7}{2}X\zeta_3-\frac{191}{72}X\pi^2+\frac{31}{2}Y\zeta_3-\frac{31}{12}\zeta_3-\frac{139}{360}\pi^4\Bigg)\frac{t^2}{u^2}+\\
&+\Bigg(\frac{5}{2}\mathrm{Li}_4(x)-\frac{5}{2}\mathrm{Li}_4(y)+8\mathrm{Li}_4(z)-\frac{13}{2}\mathrm{Li}_3(x)X+6\mathrm{Li}_3(x)Y-\frac{31}{6}\mathrm{Li}_3(x)-\\
&-6\mathrm{Li}_3(y)X+\frac{13}{2}\mathrm{Li}_3(y)Y+\frac{31}{6}\mathrm{Li}_3(y)+\frac{5}{4}\mathrm{Li}_2(x)X^2+\frac{31}{6}\mathrm{Li}_2(x)X+\\
&+\frac{11}{3}\mathrm{Li}_2(x)\pi^2+4\mathrm{Li}_2(y)XY-\frac{5}{4}\mathrm{Li}_2(y)Y^2-\frac{31}{6}\mathrm{Li}_2(y)Y+\frac{19}{48}X^4-\\
&-\frac{17}{12}X^3Y-\frac{11}{18}X^3+3X^2Y^2+\frac{7}{4}X^2Y+\frac{5}{12}X^2\pi^2+\frac{43}{18}X^2+\frac{1}{12}XY^3-\\
&-\frac{7}{4}XY^2+\frac{5}{6}XY\pi^2+\frac{1}{2}XY+\frac{11}{2}X\zeta_3-\frac{23}{16}X\pi^2+\frac{19}{27}X-\frac{1}{16}Y^4-\\
&+\frac{11}{18}Y^3+\frac{1}{4}Y^2\pi^2-\frac{26}{9}Y^2-\frac{11}{2}Y\zeta_3+\frac{23}{16}Y\pi^2-\frac{19}{27}Y-\frac{41}{180}\pi^4-\frac{1}{4}\pi^2\Bigg)\frac{t}{s}+\\
&+\frac{3}{2}Y^2\frac{s}{t}+\\
&+\Bigg(\Bigg(\frac{9}{2}\mathrm{Li}_3(x)+5\mathrm{Li}_3(y)-4\mathrm{Li}_2(x)X+\frac{53}{12}\mathrm{Li}_2(x)-\frac{5}{2}\mathrm{Li}_2(y)Y+\frac{1}{6}X^3-\\
&\quad-\frac{5}{4}X^2Y+\frac{1}{4}X^2+\frac{19}{4}XY-\frac{1}{2}X\pi^2+\frac{11}{6}X-\frac{1}{4}Y^3-\frac{19}{24}Y^2-\frac{3}{2}Y\pi^2+\\
&\quad+\frac{149}{12}Y-\frac{1}{2}\zeta_3-\frac{187}{144}\pi^2-\frac{211}{108}\Bigg)+\\
&+\Bigg(-\frac{9}{2}\mathrm{Li}_3(x)-2\mathrm{Li}_3(y)-\frac{1}{4}\mathrm{Li}_2(x)X-\frac{61}{6}\mathrm{Li}_2(x)+\mathrm{Li}_2(y)Y+\\
&\quad+\frac{7}{8}X^3-X^2+\frac{7}{3}XY-\frac{1}{3}X\pi^2+\frac{19}{3}X+\frac{17}{4}Y+\frac{9}{2}\zeta_3-\frac{5}{9}\pi^2-\frac{5}{6}\Bigg)\frac{t}{u}+\\
&+\Bigg(-12\mathrm{Li}_3(x)+2\mathrm{Li}_3(y)-\frac{1}{2}\mathrm{Li}_2(x)X-\frac{31}{12}\mathrm{Li}_2(x)-\mathrm{Li}_2(y)Y-\\
&\quad-\frac{1}{4}X^3+\frac{11}{8}X^2+\frac{2}{3}XY-\frac{1}{6}X\pi^2+\frac{15}{4}X+12\zeta_3+\frac{31}{72}\pi^2\Bigg)\frac{t^2}{u^2}+\\
&+\Bigg(-\frac{1}{2}\mathrm{Li}_3(x)+\frac{1}{2}\mathrm{Li}_3(y)-\frac{3}{2}\mathrm{Li}_2(x)X+\frac{31}{3}\mathrm{Li}_2(x)+\frac{3}{2}\mathrm{Li}_2(y)Y+\frac{5}{12}X^3-\\
&\quad-\frac{5}{4}X^2Y-\frac{5}{6}X^2+\frac{5}{4}XY^2+\frac{31}{6}XY+X\pi^2-\frac{5}{6}X-\frac{5}{12}Y^3+\\
&\quad+\frac{5}{6}Y^2-Y\pi^2+\frac{5}{6}Y-\frac{31}{36}\pi^2\Bigg)\frac{t}{s}+3Y\frac{s}{t}\Bigg)i\pi
\end{aligned}
\tag{A.37}
$$

$$
I^{gg,(2),[3]}_{s+-+-}=\Bigg(-2\mathrm{Li}_4(x)-\frac{5}{2}\mathrm{Li}_4(y)+\mathrm{Li}_3(x)X+\frac{1}{2}\mathrm{Li}_3(x)Y-\frac{3}{2}\mathrm{Li}_3(x)+\frac{1}{2}\mathrm{Li}_3(y)X+
$$

$$
\begin{aligned}
&+\frac{3}{2}\mathrm{Li}_3(y)Y+\frac{9}{4}\mathrm{Li}_3(y)+\frac{3}{2}\mathrm{Li}_2(x)X-\frac{1}{6}\mathrm{Li}_2(x)\pi^2-\frac{1}{4}\mathrm{Li}_2(y)Y^2-\\
&-\frac{9}{4}\mathrm{Li}_2(y)Y-\frac{1}{12}X^3Y+\frac{3}{8}X^2Y^2-\frac{5}{12}X^2Y+\frac{5}{8}X^2\pi^2-\frac{1}{6}XY^3-\frac{7}{6}XY^2-\\
&-\frac{1}{3}XY\pi^2+\frac{1}{2}XY-\frac{1}{2}X\zeta_3+\frac{47}{24}X\pi^2+\frac{1}{48}Y^4+\frac{13}{24}Y^2\pi^2-\frac{9}{8}Y^2-\\
&-\frac{1}{2}Y\zeta_3+\frac{17}{24}Y\pi^2+\frac{47}{12}Y-\frac{9}{4}\zeta_3+\frac{1}{36}\pi^4+\frac{3}{8}\pi^2\Big)+\\
&+\Big(-\frac{23}{4}\mathrm{Li}_4(x)-3\mathrm{Li}_4(y)-3\mathrm{Li}_4(z)+\frac{11}{2}\mathrm{Li}_3(x)X-\mathrm{Li}_3(x)Y-4\mathrm{Li}_3(x)+\\
&\quad+4\mathrm{Li}_3(y)X+\frac{1}{2}\mathrm{Li}_3(y)-\frac{9}{8}\mathrm{Li}_2(x)X^2+4\mathrm{Li}_2(x)X-\frac{11}{6}\mathrm{Li}_2(x)\pi^2-\\
&\quad-2\mathrm{Li}_2(y)XY-\frac{1}{2}\mathrm{Li}_2(y)Y-\frac{5}{32}X^4+\frac{1}{6}X^3Y-\frac{3}{4}X^2Y^2-\frac{1}{12}X^2Y+\\
&\quad+X^2\pi^2+\frac{13}{8}X^2--\frac{1}{4}XY^2-\frac{5}{6}XY\pi^2-\frac{7}{12}XY-\frac{11}{2}X\zeta_3+\\
&\quad+\frac{10}{3}X\pi^2+\frac{1}{2}X+Y\zeta_3+\frac{5}{12}Y\pi^2+\frac{133}{360}\pi^4+\frac{11}{24}\pi^2\Big)\frac{t}{u}+\\
&+\Big(-4\mathrm{Li}_4(x)-\frac{3}{2}\mathrm{Li}_4(y)-\frac{3}{2}\mathrm{Li}_4(z)+\frac{7}{2}\mathrm{Li}_3(x)X-\frac{1}{2}\mathrm{Li}_3(x)Y-\frac{9}{4}\mathrm{Li}_3(x)+\\
&\quad+2\mathrm{Li}_3(y)X-\frac{3}{4}\mathrm{Li}_2(x)X^2+\frac{9}{4}\mathrm{Li}_2(x)X-\frac{11}{12}\mathrm{Li}_2(x)\pi^2-\mathrm{Li}_2(y)XY-\\
&\quad-\frac{1}{16}X^4+\frac{1}{12}X^3Y-\frac{1}{12}X^3-\frac{3}{8}X^2Y^2+\frac{1}{3}X^2Y+\frac{3}{8}X^2\pi^2+\frac{1}{8}X^2-\\
&\quad-\frac{5}{12}XY\pi^2-\frac{7}{2}X\zeta_3+\frac{11}{8}X\pi^2+\frac{1}{2}Y\zeta_3+\frac{9}{4}\zeta_3+\frac{71}{360}\pi^4\Big)\frac{t^2}{u^2}+\\
&+\Big(\frac{1}{2}\mathrm{Li}_4(x)-\frac{1}{2}\mathrm{Li}_4(y)-\frac{1}{2}\mathrm{Li}_3(x)X+\frac{1}{2}\mathrm{Li}_3(y)Y+\frac{1}{4}\mathrm{Li}_2(x)X^2-\\
&\quad-\frac{1}{3}\mathrm{Li}_2(x)\pi^2-\frac{1}{4}\mathrm{Li}_2(y)Y^2-\frac{1}{48}X^4+\frac{1}{12}X^3Y+\frac{1}{12}X^2\pi^2+\frac{1}{2}X^2-\\
&\quad-\frac{1}{12}XY^3-\frac{1}{6}XY\pi^2+XY-\frac{23}{6}X+\frac{1}{48}Y^4-\frac{1}{12}Y^2\pi^2-\\
&\quad-\frac{3}{2}Y^2+\frac{23}{6}Y+\frac{1}{36}\pi^4-\frac{1}{2}\pi^2\Big)\frac{t}{s}+\frac{1}{4}Y^2\frac{s}{t}+\\
&+\Big(-\frac{1}{4}X^2+\frac{1}{2}XY-\frac{1}{4}Y^2-\frac{1}{4}\pi^2\Big)\frac{t^2}{s^2}+\\
&+\Bigg(\Big(\frac{3}{2}\mathrm{Li}_3(x)+2\mathrm{Li}_3(y)+\frac{15}{4}\mathrm{Li}_2(x)-\frac{1}{2}\mathrm{Li}_2(y)Y-\frac{1}{4}X^3+\frac{1}{4}X^2Y-\frac{7}{6}X^2-\\
&\qquad-\frac{11}{12}XY+\frac{1}{4}X\pi^2+\frac{1}{2}X\frac{1}{6}-Y^3-\frac{1}{24}Y^2+\frac{1}{4}Y\pi^2\frac{7}{4}Y-\\
&\qquad-\zeta_3-\frac{1}{8}\pi^2+\frac{47}{12}\Big)+\\
&\quad+\Big(\frac{9}{2}\mathrm{Li}_3(x)+4\mathrm{Li}_3(y)-\frac{1}{4}\mathrm{Li}_2(x)X+\frac{9}{2}\mathrm{Li}_2(x)-2\mathrm{Li}_2(y)Y-\frac{5}{8}X^3-\\
&\qquad-\frac{25}{12}X^2-\frac{1}{6}XY-\frac{1}{12}X\pi^2+\frac{8}{3}X-\frac{7}{12}Y-\frac{9}{2}\zeta_3-\frac{1}{2}\pi^2+\frac{1}{2}\Big)\frac{t}{u}+\\
&\quad+\Big(3\mathrm{Li}_3(x)+2\mathrm{Li}_3(y)-\frac{1}{2}\mathrm{Li}_2(x)X+\frac{9}{4}\mathrm{Li}_2(x)-\mathrm{Li}_2(y)Y-\frac{1}{4}X^3-
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{25}{24}X^2 + \frac{2}{3}XY - \frac{1}{6}X\pi^2 + \frac{1}{4}X - 3\zeta_3 - \frac{3}{8}\pi^2\Big) \frac{t^2}{u^2} + \\
&+ \Big(-\frac{1}{2}\mathrm{Li}_3(x) + \frac{1}{2}\mathrm{Li}_3(y) + \frac{1}{2}\mathrm{Li}_2(x)X - \frac{1}{2}\mathrm{Li}_2(y)Y - \frac{1}{12}X^3 + \\
&+ \frac{1}{4}X^2Y - \frac{1}{4}XY^2 + 2X + \frac{1}{12}Y^3 - 2Y\Big)\frac{t}{s} + \frac{1}{2}Y\frac{s}{t}\Big) i\pi
\end{aligned} \tag{A.38}
$$

$$
\begin{aligned}
J^{gg,(2),[3]}_{s+-+-} = &\Big(\frac{2}{3}\mathrm{Li}_3(x) + \frac{1}{3}\mathrm{Li}_3(y) - \frac{2}{3}\mathrm{Li}_2(x)X - \frac{1}{3}\mathrm{Li}_2(y)Y - \frac{5}{24}X^2Y + \frac{1}{24}XY^2 - \\
&- \frac{29}{72}XY + \frac{5}{12}X\pi^2 - \frac{1}{9}Y^3 - \frac{13}{24}Y^2 - \frac{17}{72}Y\pi^2 + \frac{19}{72}Y - \frac{1}{3}\zeta_3 + \frac{17}{12}\pi^2\Big) + \\
&+ \Big(-\mathrm{Li}_4(x) + \mathrm{Li}_4(y) + \mathrm{Li}_4(z) + \mathrm{Li}_3(x)Y + 2\mathrm{Li}_3(y) - \frac{1}{6}\mathrm{Li}_2(x)\pi^2 - 2\mathrm{Li}_2(y)Y + \\
&+ \frac{1}{24}X^4 - \frac{1}{6}X^3Y + \frac{1}{6}X^3 + \frac{1}{4}X^2Y^2 - \frac{1}{24}X^2Y + \frac{1}{12}X^2\pi^2 - \\
&- \frac{5}{12}X^2 - XY^2 - \frac{1}{6}XY\pi^2 + \frac{1}{12}XY + \frac{3}{4}X\pi^2 + \frac{1}{2}X - Y\zeta_3 + \\
&+ \frac{1}{3}Y\pi^2 - 2\zeta_3 + \frac{7}{180}\pi^4 - \frac{5}{12}\pi^2\Big)\frac{t}{u} + \\
&+ \Big(-2\mathrm{Li}_4(x) + 2\mathrm{Li}_4(y) + 2\mathrm{Li}_4(z) + 2\mathrm{Li}_3(x)Y - \frac{1}{3}\mathrm{Li}_2(x)\pi^2 + \frac{1}{12}X^4 - \\
&- \frac{1}{3}X^3Y + \frac{1}{2}X^2Y^2 + \frac{1}{24}X^2Y + \frac{1}{6}X^2\pi^2 + \frac{1}{4}X^2 - \frac{1}{3}XY\pi^2 - \\
&- \frac{1}{12}X\pi^2 2Y\zeta_3 + \frac{7}{90}\pi^4\Big) - \frac{t^2}{u^2} + \\
&+ \Big(\frac{1}{3}\mathrm{Li}_3(x) - \frac{1}{3}\mathrm{Li}_3(y) - \frac{1}{3}\mathrm{Li}_2(x)X + \frac{1}{3}\mathrm{Li}_2(y)Y + \frac{1}{9}X^3 - \frac{1}{4}X^2Y - \\
&- \frac{29}{24}X^2 + \frac{1}{4}XY^2 + XY + \frac{47}{72}X\pi^2 + \frac{5}{18}X - \frac{1}{9}Y^3 + \frac{5}{24}Y^2 - \\
&- \frac{47}{72}Y\pi^2 - \frac{5}{18}Y - \frac{1}{2}\pi^2\Big)\frac{t}{s} - \frac{1}{2}Y^2\frac{s}{t} + \\
&+ \Big(-\frac{1}{4}X^2 + \frac{1}{2}XY - \frac{1}{4}Y^2 - \frac{1}{4}\pi^2\Big)\frac{t^2}{s^2} + \\
&+ \Big(\Big(-\frac{1}{3}\mathrm{Li}_2(x) + \frac{1}{8}X^2 - \frac{2}{3}XY - \frac{29}{72}X + \frac{5}{24}Y^2 - \frac{77}{24}Y + \frac{17}{72}\pi^2 + \frac{19}{72}\Big) + \\
&+ \Big(\mathrm{Li}_3(x) + 2\mathrm{Li}_2(x) - \frac{1}{24}X^2 - \frac{1}{12}XY - \frac{3}{4}X + \frac{1}{12}Y - \zeta_3 + \frac{1}{2}\Big)\frac{t}{u} + \\
&+ \Big(2\mathrm{Li}_3(x) + \frac{1}{24}X^2 + \frac{1}{12}XY + \frac{1}{2}X - 2\zeta_3\Big)\frac{t^2}{u^2} + \\
&+ \Big(-\frac{2}{3}\mathrm{Li}_2(x) - \frac{1}{12}X^2 - \frac{1}{3}XY + \frac{11}{36}X + \frac{1}{12}Y^2 - \frac{11}{36}Y + \frac{1}{18}\pi^2\Big)\frac{t}{s} - Y\frac{s}{t}\Big) i\pi
\end{aligned} \tag{A.39}
$$

$$
\begin{aligned}
K^{gg,(2),[3]}_{s+-+-} = &\Big(-\frac{1}{3}\mathrm{Li}_3(y) + \frac{1}{3}\mathrm{Li}_2(y)Y + \frac{1}{24}X^2Y + \frac{1}{24}XY^2 + \frac{1}{8}XY - \frac{1}{12}X\pi^2 - \frac{3}{2}Y^2 + \\
&+ \frac{7}{36}Y\pi^2 + \frac{29}{24}Y + \frac{1}{3}\zeta_3 - \frac{1}{8}\pi^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-2\mathrm{Li}_4(x)+2\mathrm{Li}_4(y)+2\mathrm{Li}_4(z)+2\mathrm{Li}_3(x)Y+\frac{1}{3}\mathrm{Li}_3(x)+4\mathrm{Li}_3(y)-\\
&\quad-\frac{1}{3}\mathrm{Li}_2(x)X-\frac{1}{3}\mathrm{Li}_2(x)\pi^2-4\mathrm{Li}_2(y)Y+\frac{1}{12}X^4-\frac{1}{3}X^3Y+\frac{1}{2}X^2Y^2+\\
&\quad+\frac{1}{12}X^2Y+\frac{1}{6}X^2\pi^2+\frac{3}{4}X^2-2XY^2-\frac{1}{3}XY\pi^2-\frac{1}{4}XY-\frac{4}{9}X\pi^2+\\
&\quad+\frac{5}{6}X-2Y\zeta_3+\frac{2}{3}Y\pi^2-\frac{13}{3}\zeta_3+\frac{7}{90}\pi^4-\frac{17}{36}\pi^2\Big)\frac{t}{u}+\\
&+\Big(-4\mathrm{Li}_4(x)+4\mathrm{Li}_4(y)+4\mathrm{Li}_4(z)+4\mathrm{Li}_3(x)Y-\frac{1}{3}\mathrm{Li}_3(x)+\frac{1}{3}\mathrm{Li}_2(x)X-\\
&\quad-\frac{2}{3}\mathrm{Li}_2(x)\pi^2+\frac{1}{6}X^4-\frac{2}{3}X^3Y+X^2Y^2+\frac{1}{24}X^2Y+\frac{1}{3}X^2\pi^2+\frac{1}{4}X^2-\\
&\quad-\frac{2}{3}XY\pi^2+\frac{7}{36}X\pi^2-4Y\zeta_3+\frac{1}{3}\zeta_3+\frac{7}{45}\pi^4\Big)\frac{t^2}{u^2}+\\
&+\Big(\frac{1}{3}\mathrm{Li}_3(x)-\frac{1}{3}\mathrm{Li}_3(y)-\frac{1}{3}\mathrm{Li}_2(x)X+\frac{1}{3}\mathrm{Li}_2(y)Y+\frac{1}{4}X^2-\frac{1}{2}XY-\\
&-\frac{5}{18}X\pi^2-\frac{1}{3}X+\frac{1}{4}Y^2+\frac{5}{18}Y\pi^2+\frac{1}{3}Y+\frac{1}{4}\pi^2\Big)\frac{t}{s}-\frac{3}{4}Y^2\frac{s}{t}+\\
&+\Bigg(\Big(-\frac{1}{3}\mathrm{Li}_2(x)+\frac{1}{24}X^2-\frac{1}{6}XY+\frac{1}{8}X-\frac{1}{8}Y^2-\frac{23}{8}Y+\frac{29}{24}\Big)+\\
&\quad+\Big(2\mathrm{Li}_3(x)+\frac{11}{3}\mathrm{Li}_2(x)+\frac{1}{4}X^2+\frac{1}{6}XY+\frac{5}{4}X-\frac{1}{4}Y-2\zeta_3+\frac{1}{18}\pi^2+\frac{5}{6}\Big)\frac{t}{u}+\\
&\quad+\Big(4\mathrm{Li}_3(x)+\frac{1}{3}\mathrm{Li}_2(x)-\frac{1}{8}X^2+\frac{1}{12}XY+\frac{1}{2}X-4\zeta_3-\frac{1}{18}\pi^2\Big)\frac{t^2}{u^2}+\\
&\quad+\Big(-\frac{2}{3}\mathrm{Li}_2(x)+\frac{1}{6}X^2-\frac{1}{3}XY-\frac{1}{6}Y^2+\frac{1}{18}\pi^2\Big)\frac{t}{s}-\frac{3}{2}Y\frac{s}{t}\Bigg)i\pi
\end{aligned}
\tag{A.40}
$$

$$
\begin{aligned}
L^{gg,(2),[3]}_{s+-+-} &= \Big(-\frac{1}{72}XY-\frac{1}{36}Y^2+\frac{5}{54}Y-\frac{5}{72}\pi^2\Big)+\Big(\frac{1}{36}X^2-\frac{5}{54}X-\frac{1}{36}Y^2+\frac{5}{54}Y\Big)\frac{t}{s}+\\
&+\Bigg(\Big(-\frac{1}{72}X+\frac{1}{24}Y+\frac{5}{54}\Big)+\Big(-\frac{1}{18}X+\frac{1}{18}Y\Big)\frac{t}{s}\Bigg)i\pi
\end{aligned}
\tag{A.41}
$$

A.3 $q(p_2,+)+\bar{q}(p_1,-)\to g(p_3,-)+g(p_4,+)$

The finite remainders are obtained by Bose symmetry from the $q(p_2,+)+\bar{q}(p_1,-)\to g(p_3,+)+g(p_4,-)$ process.

$$
\begin{aligned}
A^{gg,(2),[1]}_{s+--+} &= A^{gg,(2),[2]}_{s+-+-}(t\leftrightarrow u)\,,\\
A^{gg,(2),[2]}_{s+--+} &= A^{gg,(2),[1]}_{s+-+-}(t\leftrightarrow u)\,,\\
A^{gg,(2),[3]}_{s+--+} &= A^{gg,(2),[3]}_{s+-+-}(t\leftrightarrow u)\,,
\end{aligned}
\tag{A.42}
$$

etc.

A.4 $q(p_2,+)+\bar{q}(p_1,-)\to g(p_3,-)+g(p_4,-)$

The finite remainders are obtained by Bose symmetry from the $q(p_2,+)+\bar{q}(p_1,-)\to$

$g(p_3,+) + g(p_4,-)$ process.

$$\begin{aligned} A^{gg,(2),[1]}_{s+---} &= A^{gg,(2),[2]}_{s+-++}(t \leftrightarrow u)\,, \\ A^{gg,(2),[2]}_{s+---} &= A^{gg,(2),[1]}_{s+-++}(t \leftrightarrow u)\,, \\ A^{gg,(2),[3]}_{s+---} &= A^{gg,(2),[3]}_{s+-++}(t \leftrightarrow u)\,, \end{aligned} \tag{A.43}$$

etc.

B. Finite one-loop contributions

In this appendix we list the coefficients $A, \ldots, E$ for the finite one-loop amplitudes defined in section 4.3 for $q\bar{q} \to gg$. Expressions for quark-gluon scattering as well as the processes involving photons are available from the authors as a FORM output file.

B.1 $q(p_2,+) + \bar{q}(p_1,-) \to g(p_3,+) + g(p_4,+)$

$$\begin{aligned} A^{gg,(1),[1]}_{s+-++} &= -\frac{1}{2}\frac{s}{t} - \frac{1}{3} \\ B^{gg,(1),[1]}_{s+-++} &= -\frac{1}{2}\frac{s}{t} \\ C^{gg,(1),[1]}_{s+-++} &= \frac{1}{3} \\ A^{gg,(1),[2]}_{s+-++} &= -\frac{1}{2}\frac{t}{u} - \frac{1}{6} \\ B^{gg,(1),[2]}_{s+-++} &= -\frac{1}{2}\frac{t}{u} - \frac{1}{2} \\ C^{gg,(1),[2]}_{s+-++} &= -\frac{1}{3} \\ D^{gg,(1),[3]}_{s+-++} = E^{gg,(1),[3]}_{s+-++} &= 0\,. \end{aligned} \tag{B.1}$$

B.2 $q(p_2,+) + \bar{q}(p_1,-) \to g(p_3,+) + g(p_4,-)$

$$\begin{aligned} A^{gg,(1),[1]}_{s+-+-} &= \left(-\frac{1}{2}X^2 + \frac{11}{6}X + \frac{7}{2}\right) + \left(-\frac{1}{2}X + \frac{1}{2}\right)\frac{t}{u} + X\frac{t^2}{u^2} + \frac{1}{2}X^2\frac{t^3}{u^3} + \\ &\quad + \left(\left(-X - \frac{11}{6}\right) - \frac{1}{2}\frac{t}{u} + \frac{t^2}{u^2} + X\frac{t^3}{u^3}\right)i\pi \end{aligned} \tag{B.2}$$

$$\begin{aligned} B^{gg,(1),[1]}_{s+-+-} &= \left(\frac{1}{2}X^2 + \frac{3}{2}X - \frac{7}{2}\right) + \left(\frac{3}{2}X^2 + \frac{5}{2}X + \frac{1}{2}\right)\frac{t}{u} + \\ &\quad + \left(\frac{3}{2}X^2 + X\right)\frac{t^2}{u^2} + \frac{1}{2}X^2\frac{t^3}{u^3} + \\ &\quad + \left(\left(X + \frac{3}{2}\right) + \left(3X + \frac{5}{2}\right)\frac{t}{u} + \left(3X + 1\right)\frac{t^2}{u^2} + X\frac{t^3}{u^3}\right)i\pi \end{aligned} \tag{B.3}$$

$$C^{gg,(1),[1]}_{s+-+-} = -\frac{1}{3}X + \frac{1}{3}i\pi \tag{B.4}$$

$$A^{gg,(1),[2]}_{s+-+-} = \left(-Y^2 + \frac{10}{3}Y + 3\right) - \frac{1}{2}Y^2\frac{s}{t} + \left(\left(-2Y - \frac{1}{3}\right) - Y\frac{s}{t}\right)i\pi \tag{B.5}$$

$$B^{gg,(1),[2]}_{s+-+-} = -4 - \frac{1}{2}Y^2\frac{s}{t} - Yi\pi\frac{s}{t} \tag{B.6}$$

$$C^{gg,(1),[2]}_{s+-+-} = -\frac{1}{3}Y + \frac{1}{3}i\pi \tag{B.7}$$

$$\begin{aligned} D^{gg,(1),[3]}_{s+-+-} =& \left(XY - \frac{1}{2}Y^2 + \frac{5}{3}Y - \frac{1}{2}\pi^2\right) + \frac{3}{4}X^2\frac{t}{u} + \left(\frac{1}{2}X^2 - \frac{5}{3}X - \frac{1}{2}Y^2 + \frac{5}{3}Y\right)\frac{t}{s} + \\ &+ \left(\left(X + \frac{5}{3}\right) + \frac{3}{2}X\frac{t}{u} + \left(X - Y\right)\frac{t}{s}\right)i\pi \end{aligned} \tag{B.8}$$

$$E^{gg,(1),[3]}_{s+-+-} = -\frac{1}{6}Y - \frac{1}{6}i\pi + \left(\frac{1}{6}X - \frac{1}{6}Y\right)\frac{t}{s} \tag{B.9}$$

B.3 $q(p_2,+) + \bar{q}(p_1,-) \to g(p_3,-) + g(p_4,+)$

The finite remainders are obtained by Bose symmetry from the $q(p_2,+) + \bar{q}(p_1,-) \to g(p_3,+) + g(p_4,-)$ process.

$$\begin{aligned} A^{gg,(1),[1]}_{s+--+} &= A^{gg,(1),[2]}_{s+-+-}(t \leftrightarrow u)\,, \\ A^{gg,(1),[2]}_{s+--+} &= A^{gg,(1),[1]}_{s+-+-}(t \leftrightarrow u)\,, \\ A^{gg,(1),[3]}_{s+--+} &= A^{gg,(1),[3]}_{s+-+-}(t \leftrightarrow u)\,, \end{aligned} \tag{B.10}$$

etc.

B.4 $q(p_2,+) + \bar{q}(p_1,-) \to g(p_3,-) + g(p_4,-)$

The finite remainders are obtained by Bose symmetry from the $q(p_2,+) + \bar{q}(p_1,-) \to g(p_3,+) + g(p_4,-)$ process.

$$\begin{aligned} A^{gg,(1),[1]}_{s+---} &= A^{gg,(1),[2]}_{s+-++}(t \leftrightarrow u)\,, \\ A^{gg,(1),[2]}_{s+---} &= A^{gg,(1),[1]}_{s+-++}(t \leftrightarrow u)\,, \\ A^{gg,(1),[3]}_{s+---} &= A^{gg,(1),[3]}_{s+-++}(t \leftrightarrow u)\,, \end{aligned} \tag{B.11}$$

etc.

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